MASERES, F.

~~BERNOULLI (JACQUES)~~

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MR. JAMES BERNOULLI'S

DOCTRINE

OF

PERMUTATIONS AND COMBINATIONS,

AND

SOME OTHER USEFUL MATHEMATICAL TRACTS.

THE DOCTRINE OF PERMUTATIONS AND COMBINATIONS,

BEING

AN ESSENTIAL AND FUNDAMENTAL PART

OF THE

DOCTRINE OF CHANCES;

As it is delivered by Mr. JAMES BERNOULLI, in his excellent Treatiſe on the Doctrine of Chances, intitled, *Ars Conjectandi*, and by the celebrated Dr. JOHN WALLIS, of Oxford, in a Tract intitled from the Subject, and publiſhed at the end of his Treatiſe on Algebra:

In the former of which Tracts is contained,

A Demonſtration of Sir ISAAC NEWTON's famous BINOMIAL THEOREM, in the Caſes of Integral Powers, and of the Reciprocals of Integral Powers.

TOGETHER WITH

SOME OTHER USEFUL MATHEMATICAL TRACTS.

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E R R A T A.

IN THE PREFACE.

In page xv, line 14 from the bottom, inſtead of *plaee*, read *place*.

IN THE BOOK.

In page 5, line 20, inſtead of *indicunt*, read *inducunt*.

In page 7, line 13 from the bottom, inſtead of *Auctorum*, read *Actorum*.

In page 21, line 6, inſtead of 3)2(2 : 2, read 3)3(2 : 2.

And in the ſame page 21, line 11, inſtead of 3)2(4 : 3, read 3)4(4 : 3.

In page 23, line 6 from the bottom, inſtead of *tabulà*, read *tabulâ*.

In page 25, line 16, inſtead of $\frac{n-3.e}{r}$, read $\frac{\overline{n-3}.c}{r}$.

In page 27, line 5, after the word "primam" dele the figure of 1.

And in the ſame page 27, line the 3d from the bottom, inſtead of $\frac{n.n-1}{1.2}$, read $\frac{n.\overline{n-1}}{1.2}$.

In page 28, line 10, after the word "ſubquintuplum," inſert a comma.

And in the ſame page 28, line 15, inſtead of $\frac{n.n-1}{1.2}$, read $\frac{n.\overline{n-1}}{1.2}$.

In page 30, the bottom line, inſtead of $-\frac{1}{2}nn$, read $+\frac{1}{2}nn$.

In page 32, in the laſt line but one of the lines that are parallel to the ſide of the page, inſtead of $-\frac{1}{12}nn$, read $-\frac{3}{20}nn$.

In page 50, line 2 from the bottom, inſert the mark " after the word "Mathematicks."

In page 69, line 21, inſtead of *of rows*, read *or rows*.

In page 72, in the note at bottom, inſtead of *Alterations*, read *Alternations*.

In page 73, line 6, inſtead of 252, 462, read 210, 330.

In page 87, dele the figure of 1 at the end of the firſt line.

In page 101, line 3, inſtead of 71, read 74.

In page 102, line 12, inſtead of 71, read 74.

In page 103, line 12, inſtead of $\frac{n \times f+c+\overline{d+c+b+a}}{r}$, read $\frac{n \times \overline{f+c+d+c+b+a}}{r}$.

In page 112, line 16, instead of $\frac{n.\ n-1.\ \overline{n-2}.\ \overline{n-3}}{2\times 3\times 4}$, read $\frac{n.\ \overline{n-1}.\ \overline{n-2}.\ \overline{n-3}}{2\times 3\times 4}$.

And in the ſame page 112, line 17, inſtead of $\frac{n.\ \overline{n-1}.\ \overline{n-2}.\ \overline{n-3}.\ n-4}{2\times 3\times 4\times 5}$, read $\frac{n.\ \overline{n-1}.\ \overline{n-2}.\ \overline{n-3}.\ \overline{n-4}}{2\times 3\times 4\times 5}$.

In page 117, line 11 from the bottom, inſtead of $Ba^{1-2}b^{1}$, read $Ba^{1-2}b^{2}$.

In page 144, line 7 from the bottom, inſtead of $-56x$, read $-56x^5$.

In page 151, in the bottom line, inſtead of $-\frac{n}{\ }Ax$, read $-\frac{n}{1}Ax$.

In page 158, line 3, inſtead of $21x^5+36x^6$, read $+21x^5+28x^6+36x^7$.

In page 183, lines 6, 7, and 8, the figure of 2 is not clear in the powers of 12 in the numerators of the fractions $\frac{\overline{12}|^7}{2}$, $\frac{12\times\overline{12}|^7}{8}$, and $\frac{7\times\overline{12}|^3}{2}$.

In page 184, line 5 from the bottom, inſtead of 116,12, read 116,122.

In page 196, line 8 from the bottom, inſtead of $\frac{11}{11}$, read $\frac{n^{11}}{11}$.

In page 197, the top line, inſtead of $\frac{5n}{6}$, read $\frac{5n}{66}$.

In page 218, line 3, inſtead of

$$+nBx^{n-1}d^2+n\times\frac{n-1}{2}Bx^{n-2}d^3+n\times\frac{n-1}{2}\times\frac{n-2}{3}Bx^{n-3}d^4, \&c$$

read $+nBx^{n-1}d+n\times\frac{n-1}{2}Bx^{n-2}d^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}Bx^{n-3}d^3$, &c.

And in the ſame page 218, line 4, inſtead of

$$+n-1\times Cx^{n-2}d^3+n-1\times\frac{n-2}{2}Cx^{n-3}d^4, \&c, \text{ read}$$

$$+\overline{n-1}\times Cx^{n-2}d+\overline{n-1}\times\frac{n-2}{2}\times Cx^{n-3}d^2, \&c.$$

And again in the ſame page 218, line 5, inſtead of $+n-2\ Dx^{n-3}d^4$, &c, read $+\overline{n-2}\times Dx^{n-3}d$, &c.

In page 221, line 5 from the bottom, inſtead of $\frac{n\times n-1\times n-2}{2.3.4.5.6}$, read $\frac{n\times\overline{n-1}\times\overline{n-2}}{2.3.4.5.6}$.

And

And in the ſame page 221, line 2 from the bottom, inſtead of $\frac{n \times n-1 \times n-2\, x^{n-3}}{2.3.4.5.6}$, read $\frac{n \times \overline{n-1} \times \overline{n-2} \times x^{n-3}}{2.3.4.5.6}$.

And again in the ſame page 221, the bottom line, inſtead of $\frac{n \times n-1 \times n-2}{2.3.4.5.6}$, read $\frac{n \times \overline{n-1} \times \overline{n-2}}{2.3.4.5.6}$.

In page 239, line 11, inſtead of a_m, read a^m.

In page 264, line 6 from the bottom, inſtead of $\overline{1+1}|$, read $\overline{1+1}|^n$.

In page 298, line 6, inſtead of *bc*, read *bbc*.

In page 302, line 10 from the bottom, the firſt figure after 10, which ſhould be a 3, is not clear.

In page 340, line 16 from the bottom, inſtead of 391, read 191.

In page 341, line 15 from the bottom, inſtead of 3, 67, read 31, 67.

In page 350, line 11, inſtead of 92 × 76, read 29 × 67.

In page 369, line 9 from the bottom, column 17, inſtead of p, read 7.

In page 371, line 6 from the bottom, column 14, inſtead of 19, read 37.

In page 374, line 11 from the bottom, column 10, inſtead of 73, read 107.

In page 435, line 9 from the bottom, inſtead of 3, read 2.

In page 446, line 13 from the bottom, inſtead of $\frac{4m^2n^3}{\overline{mm+nn}|^2}$, read $\frac{4m^2n^2}{\overline{mm+nn}|^2}$.

In pages 470, 471, 472, and 473, the title at the top of the pages is wrong. It ſhould be, *Of the Extraction of the Cube-root by Mr. de* Lagny'*s Method of Approximation.*

In page 472, lines 8 and 9 from the bottom, inſtead of 3.264, read 0.3264.

In page 488, line 4, after the letter *a*, inſert the firſt mark of a parentheſis, to wit (.

In page 495, line 7, inſtead of $\frac{a^3 - c \times a}{c + 2a^3}$, read $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$.

In page 511, line 4 from the bottom, inſtead of $- Ca^{m-2}z^2$, read $+ Ca^{m-2}z^2$.

And in the ſame page 511, line 2 from the bottom, inſtead of $+ Ca^{m-2}$, read $+ Ca^{m-2}z^2$.

In page 540, lines 10 and 15, inſtead of $\sqrt{}^3$, read $\sqrt{}^m$.

In

ERRATA.

In page 542, line 9 from the bottom, instead of $\frac{2a \times a^{m} - \mathrm{N}}{\overline{m-1} \times \mathrm{N} + \overline{m+1} \times a^{m}}$,

read $\frac{2a \times \overline{a^{m} - \mathrm{N}}}{\overline{m-1} \times \mathrm{N} + \overline{m+1} \times a^{m}}$.

In page 561, in the Note, line 19 from the bottom, instead of $- b^{z}x^{3}$, read $- b^{2}x^{3}$.

THE

PREFACE.

IT is well known to persons acquainted with the hiſtory of the Mathematicks, that Sir Iſaac Newton's celebrated Theorem concerning the powers of a binomial quantity, ſuch as $a+b$, was communicated by him to the world, in the latter part of the laſt century, without a demonſtration. And many other writers on Algebra ſince Sir Iſaac's death, and, amongſt the reſt, the famous Profeſſor *Saunderſon*, of the Univerſity of Cambridge, have followed his example in taking this important Theorem for granted, and delivering it to their readers without attempting to demonſtrate it. For the chapter on this ſubject in the ſecond volume of the Profeſſor's *Elements of Algebra*, in two volumes, quarto, contains nothing more than a full and clear deſcription of the Theorem, with an application of it to a good number of well-choſen examples, by way of illuſtration. This neglect of demonſtrating ſo important a propoſition has always appeared to me very ſtrange; as the great merit and glory of the mathematical Sciences conſiſts in the certainty of the principles on which they are founded, and the clearneſs and regularity with which all the ſubſequent concluſions obtained in them are deduced from thoſe fundamental principles. There have been, however, other eminent Mathematicians who have ſupplied this great omiſſion, and given us juſt and accurate demonſtrations of this Theorem in ſome of the more obvious and important cafes of it, though not, perhaps, in all its cafes. And of thefe I confider Mr. *James Bernoulli* (who was Profeffor of Mathematicks at Bafil, or Bafle, in Switzerland, in the latter part of the laft century,) as one of the moft fuccefsfull. For, in the 3d chapter of the fecond part of his excellent Treatife on the Doctrine of Chances, intitled, *Ars Conjectandi*, (which was publifhed at Bafil in the year 1713, in a fmall quarto volume, fome years after his death,) there is a demonftration of this celebrated Theorem in the firft, or fimpleft, cafe of it, (or when m, or the index of the power of the binomial quantity $a + b$, is an affirmative whole number,) that is deduced from the very nature of Multiplication and the properties of the Figurate numbers, in the cleareft and moft accurate manner poffible. So that no demonftration of it ought to be expected, or need be defired, that fhall exceed, or even equal, this in point of accuracy and perfpicuity, though fome others may, perhaps, be fomewhat fhorter. This demonftration I was therefore defirous of making more generally known to the Students of the Mathematicks; and with that view I refolved to re-publifh it, together with fo much of the concomitant text of Mr. Bernoulli's faid valuable Treatife, as was neceffary to the thorough underftanding of it, in a volume of a moderate fize and price. This was the inducement that gave rife to the prefent publication.

To anfwer this purpofe in the moft effectual manner, I thought it would be beft to re-publifh the whole of the three firft chapters of the fecond Part of the faid Treatife of Mr. James Bernoulli, together with the Preface to the faid fecond Part; but without the firft Part of the fame work, becaufe the faid firft part, (though in itfelf important and curious, and effential, I doubt not, to the full underftanding of *the Doctrine of Chances*,) is not at all neceffary to the underftanding of the fecond Part, which treats of the Doctrine of Permutations and Combinations, and begins, in the moft diftinct and elementary manner, with the firft foundations of that doctrine. And further, as there are many perfons in England that are fond of the Mathematical Sciences without

without having much acquaintance with the Latin language, I have, in order to render the contents of these three valuable chapters accessible to such persons, translated these chapters into English, and subjoined the translation to the original text in Latin; so that the reader may chuse in which of the two languages he will peruse them. And in this translation I have expressed myself in a fuller manner than Mr. Bernoulli had adopted in the original, because I had observed that the great degree of brevity with which Mr. Bernoulli had expressed himself had rendered some parts of the original rather obscure. And I have likewise added a few notes both to the original and the translation, where the text seemed to me to require them.

And further, in the latter part of the translation of these chapters, I have also done something more than merely translate them. For, as I observed that Mr. Bernoulli's conclusions concerning the properties of the Figurate numbers, (which he had applied to the demonstration of the binomial theorem in the first, or simplest, case of it, or when m, or the index of the power of the binomial quantity $a + b$, was *an affirmative whole number*,) might easily be applied to the demonstration of the binomial theorem in another case of it, to wit, in that case of it in which m, or the index of the binomial quantity $a + b$, is *a negative whole number*, I drew up some additional articles, that are not contained in Mr. Bernoulli's text, for this purpose. These additional articles, (which contain a demonstration of the binomial theorem in *the case of integral and negative powers*, or in the case of the quantity $\overline{a + b}|^{-m}$,) extend from page 123 to page 166; after which the translation of Mr. Bernoulli's text is resumed, and continues to page 213.

These three chapters contain a most accurate and distinct explanation of the fundamental parts of the Doctrine of Permutations and Combinations, and of the most remarkable properties of the Figurate numbers, which, it is well known, are of the most extensive use in various branches of the Mathematicks. And they likewiſe contain an application of the properties of theſe important numbers to the ſummation of the ſquares of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to any propoſed number n, and to the ſummation of the cubes, and of the fourth powers, and of the fifth powers, and of all the following powers, of the ſame numbers, (which is a matter of much nicety and difficulty, and was formerly a great object of inquiry to Mathematicians,) as well as to the demonſtration of the binomial theorem in the caſe of integral and affirmative powers, and (with the articles I have added to it in pages 123, &c, to 166,) in the caſe of integral and negative powers. All which, together, makes a conſiderable body of very uſeful mathematical learning.

Immediately after theſe three chapters of Mr. James Bernoulli's *Ars Conjectandi*, I have re-publiſhed the tenth Mathematical Eſſay of the late very learned and ingenious Mathematician, Mr. Thomas Simpſon, of Woolwich Academy, which is a ſolution of the following Problem, to wit, "*To find the ſum of any ſeries of powers whoſe roots are in "arithmetical progreſſion, as* $\overline{m + d}|^{n}$, $\overline{m + 2d}|^{n}$, $\overline{m + 3d}|^{n}$, "$\overline{m + 4d}|^{n}$, $\overline{m + 5d}|^{n}$, - - - - - - x^{n}, *the letters* m, d, *and* "n, *denoting any numbers whatſoever.*" This Eſſay of Mr. Simpſon had been alluded to in a note to the tranſlation of the foregoing extract from Mr. Bernoulli's book, at the bottom of page 213; and it is ſo nearly connected with the ſubject of the latter part of that extract relating to the ſums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, that I thought it would be agreeable to the reader to have it laid before him immediately after the ſaid extract; and therefore I cauſed it to be reprinted in that place. It extends from page 214 to page 224.

The next Tract is one of my own compoſition, and contains *An Inveſtigation and Demonſtration of Sir Iſaac Newton's*

Binomial

Binomial Theorem in the case of integral and affirmative powers; in which the law of the generation of the numeral co-efficients of the terms of the series which is equal to the quantity $\overline{a+b}|^m$, *is discovered by a conjecture grounded on the observation of the law of the said co-efficients in some particular examples; but, when so discovered, is shewn to be true universally in all other integral and affirmative powers whatsoever of the said binomial quantity, by a strict and accurate demonstration.* This Tract begins in page 227, and ends in page 268, and contains, as I believe, the best and most satisfactory demonstration of the Binomial Theorem in the case of integral and affirmative powers that has yet been given of it, next to that contained in the foregoing Extract from Mr. James Bernoulli's Treatise, intitled *Ars Conjectandi.* The conjectural investigation of the law of the numeral co-efficients of the terms of the series that is equal to $\overline{a+b}|^m$, given in this Tract, is suggested by Professor Saunderson, in the second volume of his Algebra, in the chapter on the Binomial Theorem; where (as I before observed,) the reader will find a good explanation and illustration of the said celebrated Theorem by a variety of examples, both in the case of integral powers and in the case of roots and other fractional powers, and even in the case of negative powers and of powers that are both fractional and negative; but no demonstration of it in any case, not even in that of integral and affirmative powers. And the following strict demonstration of this Theorem in the case of integral and affirmative powers, (which begins in page 252, and ends with page 264,) is nearly the same with that which is given by Mr. John Stewart, of Aberdeen, in the 6th section of his Commentary on Sir Isaac Newton's curious little Tract, intitled, *Analysis per Æquationes numero terminorum infinitas*, or *Analysis by Equations of an infinite number of Terms.* See his edition of Newton's Treatise *on the Quadrature of Curves*, and of the said Tract, intitled, *Analysis, &c*, with his learned comments on both, in one volume, quarto, published at London in the year 1745, page 471, art. 155.

This

This Tract, concerning the said conjectural investigation and subsequent general demonstration of the Binomial Theorem in the case of integral and affirmative powers, contains the substance of two Tracts published in the year 1792, in the second volume of the Collection of Mathematical Tracts, in quarto, called, *Scriptores Logarithmici*; to wit, the 15th Tract, which extends from page 153 to page 169, of the said second volume, and the 23d, or last, Tract in the said volume, which extends from page 587 to page 591.

Next to this Tract on the Investigation and Demonstration of Sir Isaac Newton's Binomial Theorem, I have republished a Tract of the learned Dr. John Wallis, of Oxford, on the same Doctrine of Permutations and Combinations, which is the subject of the foregoing Extract from Mr. James Bernoulli's work above-mentioned. This Tract was published with Dr. Wallis's Algebra in the year 1685, under the title of *A Discourse of Combinations, Alternations, and Aliquot Parts*, and is mentioned by Mr. James Bernoulli in the foregoing Extract of his *Ars Conjectandi*, in the Scholium in pages 29 and 166, as a well-known and valuable Treatise on the properties of the Figurate numbers. And it does indeed contain a great deal of excellent and curious matter concerning those numbers, and the other subjects of which it treats, but without that accuracy and regularity in the manner of deducing the conclusions of it one from another, which distinguish the foregoing chapters of Mr. Bernoulli's work. However, on account of its intrinsick merit, and its relating to the same subjects, in a great measure, as the said Extract from Mr. Bernoulli's book, I thought it would be agreeable to my readers to see a re-publication of it in the same volume with the said Extract, and therefore I have given it a place in this Collection. It begins in page 271, and extends to page 351.

Much of this Discourse of Dr. Wallis relates to *Prime*, or *Incomposit*, numbers, and to curious arithmetical questions depending on them. And in one part of it, to wit, in page 318, the Doctor speaks of the great convenience of having

at

at hand a Table of Prime Numbers ſet down in regular order, to be referred to when we want to know into what prime numbers a given odd number may be reſolved. And he mentions a very uſeful Table of this kind that had been drawn up by a Mr. *Thomas Brancker*, M. A. and publiſhed by him in the year 1668, in an Appendix to an Engliſh tranſlation, made by him, of *Rhonius*'s *Algebra*, which had been publiſhed in the German language at Zurich in Switzerland, in the year 1659, under the title of *Algebra Rhonii, Germanicè*. This Engliſh tranſlation of Rhonius's Algebra was publiſhed by Mr. Brancker under the inſpection, and with the aſſiſtance, of Dr. John Pell, an eminent Mathematician in the reign of King Charles the Second, and ſome conſiderable additions were made to the tranſlation by Dr. Pell himſelf; which has given occaſion to the book's being ſometimes ſpoken of by ſubſequent writers of Mathematicks, and amongſt others by Dr. Wallis himſelf in this Diſcourſe, page 319, by the name of *Dr. Pell's Algebra*.

This Table of Prime Numbers Dr. Wallis ſet a high value on, inſomuch that he took the pains to examine it carefully throughout, and to correct the few errors that he found in it; ſo that now, with his corrections, it may be conſidered as very accurate. This Table therefore, together with the Appendix in which it is contained, I have here cauſed to be re-printed immediately after the foregoing Diſcourſe of Dr. Wallis. It contains not only all the Prime numbers that are leſs than 100,000, but all the odd numbers whatſoever that are leſs than that number, (except ſuch odd numbers as end with the figure of 5, and are therefore evidently diviſible by the number 5,) and it diſtinguiſhes the Prime numbers from the other odd numbers, by annexing to them the letter p; and it annexes likewiſe to every other odd number (that is not a Prime, or Incompoſit, number, but is the product of the multiplication of two, or more, leſſer numbers,) the leaſt of the prime numbers into which it may be reſolved. This Appendix, with the ſaid Table of odd numbers contained in it, extends from page 353 to page 416.

The next Tract in this Collection relates to the Rational Numbers that will exprefs the Sides of Right-angled Triangles, and contains two methods of finding as many fets of numbers as we pleafe that fhall have this property. The firft of thefe methods begins in page 417, and ends in page 431, and the fecond reaches from page 431 to page 448; after which I have inferted a Table of the Squares of the feveral natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, as far as 100, together with two additional columns adjoining to the column of the faid fquares, in the former of which I have fet down the differences of the faid fquares, and in the latter the differences of thofe differences, or the fecond differences of the fquares themfelves; which fecond differences are all equal to each other, and to the number 2. This Table begins in page 449, and is accompanied with fome remarks which extend to page 457. This Tract has a confiderable refemblance to fome parts of the foregoing Difcourfe of Dr. Wallis, and may afford fome amufement to fuch readers as are fond of contemplating the properties of numbers.

The next Tract relates to the Cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, and to the differences of the faid cubes, and the differences of the faid differences, or the fecond differences of the faid cubes themfelves, and to the differences of the faid fecond differences, or the third differences of the cubes; which third differences are all equal to each other, and to the number 6. And in pages 460, 461, and 462, I have exhibited a Table of the Cubes of all the faid natural numbers as far as 100, together with the 1ft, 2d, and 3d differences of the faid cubes in adjoining columns; after which follows an extract from a learned letter of the celebrated Mr. Leibnitz to Mr. Oldenburgh, the Secretary of the Royal Society of London, dated from London on the 3d day of February, 1672-3, relating to the fubject of the differences of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, and to the feveral fucceffive orders of fuch differences, and to the ultimate equality

lity of the feveral fucceffive differences in the fecond, or third, or fourth, or fifth, or other fubfequent, order of the faid differences, according to the height of the power to which the faid numbers are raifed, and relating to other curious properties of numbers. This letter is in Latin, and extends from page 463 to page 469, and is re-printed from the celebrated *Commercium Epiftolicum* of Mr. John Collins, and other eminent Mathematicians of the latter part of the laft century, that was firft printed by the order of the Royal Society in the year 1712, and was afterwards re-printed in the year 1722. The remaining part of this Tract, from page 469 to page 504, relates to Monfieur de Lagny's Method of Extracting the Cube-roots of Numbers by Approximation, and begins with fhewing the ufefulnefs of the foregoing Table of Cubes, in finding the firft near value of the cube-root fought, to one, or two, places of figures, which is to be the bafis of a further approximation to it by Mr. de Lagny's method. The four expreffions given by Mr. de Lagny for the fecond near value of the cube-root that is fought, are ftated in pages 470 and 471; and in the following pages to page 483, examples are given of the extraction of the cube-roots of the three numbers 2, 231, and 37,945, and of the long number 696,536,483,318,640,035,073,641,037, to a great degree of exactnefs, by means of fome of the faid expreffions, after the firft near values of the faid cube-roots to one or two places of figures, have been obtained by the help of the foregoing table of the cubes of the firft hundred numbers. Thefe four examples (in which the feveral proceffes are ftated very much at length,) will, I apprehend, be fufficient to make the reader familiarly acquainted with the method of ufing the faid expreffions of Mr. de Lagny in the extraction of the cube-roots of numbers, and at the fame time to convince him of the great ufefulnefs of thefe expreffions for effecting that purpofe; and they will likewife fhew the ufefulnefs of the foregoing Table of Cubes, in obtaining the firft near values of the cube-roots fought, to one, or two, places of figures, from which the more accurate values of them are afterwards derived by means of Mr. de Lagny's expreffions. After thefe four examples of Mr. de

Lagny's method of Approximation, follows a Scholium (in pages 484, 485, and 486,) concerning the invention of theſe expreſſions, and of Mr. Raphſon's and Sir Iſaac Newton's methods of extracting the Cube-roots, and other higher roots, of given numbers, and even the roots of affected equations of any order, by ſimilar approximations; which methods were invented by thoſe eminent Mathematicians before the publication of theſe expreſſions of Mr. de Lagny. And, then, (in pages 486, 487, 488, &c, to page 500,) I have given very full and accurate inveſtigations of the foregoing expreſſions of Mr. de Lagny, which had been only ſtated in pages 470 and 471, and illuſtrated by examples in the following pages, from page 471 to page 483. And, laſtly, in pages 501, 502, 503, 504, I have given a further illuſtration of the ſaid expreſſions of Mr. de Lagny, by applying ſome of them to the extraction of ſome of the cube-roots which had been obtained in the foregoing examples by means of others of them; with a view to make a compariſon between the different expreſſions given for the ſame purpoſe by Mr. de Lagny, and to diſcover which of them are the moſt exact, or the moſt eaſy to practice, and in which caſes it will be moſt adviſeable to reſort to ſome of them in preference to the others. This Tract (which begins in page 459, and ends in page 504,) I conſider as a very uſeful one to young ſtudents of Arithmetick and Algebra.

Having in the foregoing Tract very fully explained, and illuſtrated by examples, Mr. de Lagny's method of Extracting the Cube-roots of given numbers by Approximation, I proceed in the next Tract to ſtate his general method of Extracting any higher Roots whatſoever of given numbers by ſimilar Approximations. All theſe approximations are grounded on the ſame principle, and conſiſt in putting ſome letter of the alphabet, as a, for the known part of the root ſought, (which known part is found by conjecture, or otherwiſe, as the caſe may admit,) and putting ſome other letter, as z, for the unknown difference by which x, or the true root of the given number (which may be called N,) exceeds, or falls ſhort of, the firſt value a, (which is ſuppoſed to be known,) and then ſubſtituting $a + z$, or $a - z$, inſtead

inſtead of x, in the original equation $x^5 = N$, or $x^7 = N$, or, in general, $x^m = N$, (whereby the ſaid equation will be transformed into another equation in which z will be the only unknown quantity,) and, laſtly, in reſolving this transformed equation (of which z is the root,) as if it was only a quadratick equation, or omitting, or expunging from it, all the terms that involve any higher power of z than the ſquare. By ſuch a reſolution of this transformed equation Mr. de Lagny obtains a value of z that approaches nearly to its true value: and conſequently, by ſubſtituting this near value of z, inſtead of z, in the binomial quantity $a + z$, or $a - z$, (which is equal to x, or the root ſought, or the mth root of the given number N,) he obtains a near value of $a + z$, or $a - z$, or a ſecond near value of x, or $\sqrt[m]{N}$, which is much nearer to its true value than a, or its firſt near value, was. To explain in a full and diſtinct manner this method of extracting the mth root of any given number N, and to illuſtrate it by a few examples of the extraction of ſome high roots of given numbers, by means of the general expreſſions of the values of $a + z$ and $a - z$ derived from it, is the object of the preſent Tract. And, as the inveſtigations neceſſary for this purpoſe are very general, and, from that circumſtance, are rather more ſubtle and difficult than the inveſtigations in the preceeding Tract, (which related only to the extraction of the cube-roots of given numbers,) I have taken great pains to ſet down all the ſteps in them in regular order, as clearly and plainly as I could; which may make them appear longer than might, perhaps, have been expected, but will, in fact, enable the reader to make himſelf perfect maſter of them in leſs time than if they had been compreſſed within a narrower compaſs. The general expreſſions that are thus inveſtigated, are no leſs than four; to wit, two near values of $a + z$, obtained by conſidering the aforeſaid transformed equation (ariſing from the ſubſtitution of $a + z$, inſtead of x, in the original equation $x \quad = N$,) as a quadratick equation, and reſolving it, as ſuch, in two different manners, to wit, firſt,

 imperfectly,

imperfectly, and ſecondly, in an accurate manner; and two near values of $a - z$ obtained in like manner, by conſidering the other transformed equation, (ariſing from the ſubſtitution of $a - z$, inſtead of x, in the original equation $x^m = N$,) as a quadratick equation, and reſolving it, as ſuch, in two different manners, to wit, firſt, imperfectly, and ſecondly, in an accurate manner. In order to perform theſe inveſtigations the more eaſily and diſtinctly, I have divided the ſubject into two caſes, with Problems correſponding to them, according as x, or $\sqrt[m]{N}$, is greater, or leſs, than its firſt value a, or is equal to $a + z$, or to $a - z$. The firſt caſe, or that in which x, or $\sqrt[m]{N}$, is equal to $a + z$, is conſidered in the firſt Problem; and the ſecond caſe, or that in which x, or $\sqrt[m]{N}$, is equal to $a - z$, is conſidered in the ſecond Problem: and from the Solution of the firſt Problem we obtain the two following expreſſions, to wit, $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ and $a + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}} - \frac{a}{m-1}$, for near values of the binomial quantity $a + z$, or for ſecond near values of x, or $\sqrt[m]{N}$; and from the Solution of the ſecond Problem we obtain the two following expreſſions, to wit, $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ and $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$, for near values of the reſidual quantity $a - z$, or for ſecond near values of x, or $\sqrt[m]{N}$. And the Solutions of both theſe Problems are illuſtrated by a few ſuitable examples, placed at the end of each ſolution reſpectively, of the extraction

traction of different roots of given numbers by means of the said general expressions obtained in the preceeding solutions. The Solution of the first of these Problems begins in page 508, and ends in page 516; and is followed by three examples, which begin in page 516 and end in page 525: after which I have inserted a Scholium containing a comparison between Mr. de Lagny's aforesaid method of extracting the roots of given numbers, and Mr. Raphson's method of performing the same thing; which is somewhat simpler and easier than Mr. de Lagny's method, though not quite so exact. For the difference between the two methods consists only in this, that, whereas Mr. de Lagny resolves the transformed equation arising from the substitution of $a + z$ instead of x in the original equation $x^m = N$, as if it was a quadratick equation, omitting all the terms of it that involve any higher power of z than its square, Mr. Raphson resolves the same equation as if it was a mere simple equation, or omits all the terms of it that involve any higher power of z than its simple power, or z itself; which makes his expression of the near value of $a + z$, or of the second near value of x, or $\sqrt[m]{N}$, derived from the said transformed equation, a good deal simpler and easier to manage than those of Mr. de Lagny. This Scholium extends from page 525 to page 529, and is followed by a fourth example of the extraction of the root of a very long number by Mr. de Lagny's method, which extends to page 534. The Solution of the second of the said Problems begins in page 536, and extends to page 546, and is followed by two examples of the extraction of the roots of given numbers by means of the general expressions obtained in it, that extend from page 547 to page 554. And then the Tract concludes with some Observations, in pages 555 and 556, on the several different methods that may be taken for the extraction of the roots of numbers.

This Tract, as well as the last before it, concerning the Extraction of the Cube-roots of given Numbers, will, I hope, be found to be of great use to the Students of Arithmetick and Algebra.

The laſt Tract in this Collection is intitled, *Obſervations on Mr. Raphſon's Method of reſolving Affected Equations of all degrees by Approximation.* It begins in page 559, and ends in page 590; and its contents may be deſcribed as follows. The firſt part of it, as far as page 571, is intended, partly, to remove ſome difficulties that occur in reading Mr. Raphſon's excellent Treatiſe on the Reſolution of all Equations, (whether pure or affected,) by Approximation, intitled, *Analyſis Æquationum Univerſalis*, which difficulties are not inherent in the ſubject itſelf, or neceſſarily belonging to his method of reſolving equations, but have ariſen merely from his having unfortunately adopted the doctrine and language of *negative roots* of equations, by which the Science of Algebra, or Univerſal Arithmetick, has been diſgraced and rendered obſcure and difficult, and diſguſting to men of a juſt taſte for accurate reaſoning, ever ſince its introduction by Harriot and Des Cartes. The firſt part of this Tract is, I ſay, intended, partly, to remove ſome difficulties of this kind, in the ſaid Treatiſe of Mr. Raphſon, and, partly, to illuſtrate his method of reſolving high equations in other caſes, or where no negative roots are mentioned, by performing the reſolution of one of the equations given by him in his examples, to wit, of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, in a very full and diſtinct manner, with every ſtep of the reſolution, and the reaſonings upon which it is grounded, ſet forth at length, agreeably to the principles laid down by him in the beginning of the ſaid Treatiſe, inſtead of reſorting (as he has done in his reſolution of the ſame example, and in thoſe of all his other examples,) to the repeated application of a general theorem, or canon, that he has deduced from the ſaid principles: becauſe that way of performing the ſaid reſolution, by means of a theorem, or canon, affords much leſs ſatisfaction to the mind of the reader, or operator, in the uſe of it, than he would receive by performing the reſolution of the equation by the immediate application of the principles themſelves, as I have done, in the reſolution here given of the ſaid equation. And the following part of this Tract contains a compariſon between Mr. Raphſon's method of Reſolving Equations

Equations by Approximation, and Sir Iſaac Newton's method of Reſolving them alſo by Approximation, (which, after the firſt proceſs of the approximation, or the diſcovery of the ſecond near value of the root of the equation, differs a little from Mr. Raphſon's method,) in order to diſcover which of the two methods deſerves to be reckoned the moſt convenient. This compariſon between theſe two methods of reſolving equations by approximation, (the reſult of which is, that Mr. Raphſon's method appears to me, upon the whole, more convenient than Sir Iſaac Newton's,) reaches from page 571 to page 586: and the few remaining pages of this Tract, from page 586 to page 590, relate partly to the method of trying the exactneſs of the near value of x, or the root of the propoſed equation, which has been obtained by either of the ſaid two methods of Approximation, and, partly, to the method of finding a, or the firſt near value of x, or of the root of the propoſed equation, to a moderate degree of exactneſs, in certain difficult caſes, to wit, in thoſe caſes in which the propoſed equation either has, or (from the changes of the ſigns of its terms from + to —, and from — to +,) ſeems to have, more than one real and affirmative root.

In the next plaee I have re-publiſhed a uſeful Table of Numbers, from a book intitled *The Calculator*, publiſhed in octavo in the year 1747, by the late learned Mr. James Dodſon, being a Table of the Square-roots and Cube-roots of all the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, to 180, carried to ſeven places of figures; which may often be the means of ſaving a Student of theſe Sciences ſome time and pains in performing the calculations that may occur in them. This Table is contained in pages 591 and 592.

And in the laſt place I have re-publiſhed a Table of the Square-roots of all the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, as far as 1000, and likewiſe of the Reciprocals of all the ſaid numbers, or of the values of of the fractions $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, $\frac{1}{12}$, $\frac{1}{13}$, &c, as far as $\frac{1}{1000}$, expreſſed in decimal fractions, from the fourth volume of Dr. Charles Hutton's *Miſcellanea Mathematica*, publiſhed in the year 1775, in four little volumes, duodecimo. This Table begins in page 595, and ends in page 604, and, with Dr. Hutton's explanatory account of it in pages 605 and 606, concludes the preſent volume.

ARTIS CONJECTANDI

PARS SECUNDA,

CONTINENS

DOCTRINAM DE PERMUTATIONIBUS ET COMBINATIONIBUS.

PROOEMIUM.

INFINITAM varietatem, quæ cùm in naturæ operibus, tùm in actionibus mortalium elucet, quáeque præcipuam hujus universi pulchritudinem constituit, non aliunde quàm ex diversimodâ compositione, mixturâ & transpositione partium ejus inter se originem ducere palàm est. Sed, quia multitudo rerum ad effectum aliquem producendum concurrentium sæpenumerò tanta est támque varia, ut difficillimum sit recensere vias omnes, quibus earundem compositio, vel mixtura, fieri, vel non fieri, potest, hinc fit ut nullum sit vitium, in quod homines etiàm maximè prudentes & circumspecti frequentiùs incidant illo, quod Logici communitèr appellant *insufficientem enumerationem partium*; adeò quidem ut non verear dicere, hanc unicam ferè scaturiginem esse infinitorum, eorúmque gravissimorum, errorum, quos in ratiociniis nostris circà res tùm cognoscendas tùm agendas quotidiè committimus. Quarè merito suo utilissima censenda est ars, *combinatoria* dicta, quæ huic mentis nostræ defectui medetur, docétque sic enumerare modos omnes possibiles, ſecundùm quos res plures permiſceri, tranſponi, vel conjungi, invicèm poſſunt, ut certi ſimus, nos nullum eorum prætermiſiſſe, qui inſtituto noſtro conducere valent. Quanquam enim hoc negotii eatenùs ſit conſiderationis Mathematicæ, quatenùs in ſubducendo calculo terminatur; ſi tamen uſum & neceſſitatem ſpectes, univerſale prorſùs eſt & ita comparatum, ut ſine illo nec ſapientia Philoſophi, nec Hiſtorici exactitudo, nec Medici dexteritas, aut Politici prudentia, conſiſtere queat. Argumento ſit hoc unicum, quòd omnis horum labor in *conjectando*, & omnis conjectura in trutinandis cauſarum complexionibus aut combinationibus verſatur. Unde quoque nonnulli eximii viri, ac nominatìm Schootenius, Leibnitius, Walliſius, Preſtetus, materiam hanc ſibi tractandam ſumpſêre, ne quis exiſtimet nova eſſe hìc omnia quæ prolaturi ſumus; tametſi quædam non contemnenda de noſtro adjecimus, imprimis demonſtrationem generalem & facilem proprietatis numerorum figuratorum, cui cætera pleraque innituntur, & quam nemo, quod ſciam, ante nos dedit eruítve. Cùm itaque nondum plenum Artis ſyſtema habeamus, tùm verò, ne illa quæ habemus aliunde petere ſit opus, viſum eſt totam Doctrinam ab ovo ordiri, ac, ne quid indemonſtratum relinquatur, ex primis fundamentis eruere; quod tamèn brevitèr fiet & ſuccinctè, nec niſi in quantum inſtituti noſtri ratio exigere videtur. Totam Tractationem ad duo ſumma capita referimus, quorum unum Permutationum, alterum Combinationum doctrinam perſequitur; cui accedit tertium, quod utraſque mixtìm contemplatur.

CAPUT I.

DE PERMUTATIONIBUS.

PERMUTATIONES rerum voco variationes, juxtà quas, ſervatâ eâdem rerum multitudine, ordo ſitúſque inter ipſas diverſimodè permutatur.

Itaque ſi quæratur, quoties nonnullæ res tranſponi vel permiſceri invicèm poſſint, ſic ut ſemper accipiantur omnes ſolo

ſolo ordine ſitûve mutato, dicentur quæri omnes permutationes rerum illarum.

Res autem permutandæ vel omnes poſſunt eſſe diverſæ, vel aliquot earum eædem; quæ quidem per totidem Alphabeti literas, ſive diverſas ſive eaſdem, commodè deſignabuntur.

1. *Si res omnes permutandæ ſunt diverſæ:*

CUM numerus permutationum in rebus pluribus iniri nequeat, niſi idem priùs in omnibus aliis numero paucioribus compertus habeatur, liquet in hâc inquiſitione utendum viâ ſyntheticâ, hoc eſt, ordiendum nobis eſſe ab hypotheſibus omnium primis & ſimpliciſſimis:

Unius rei, vel literæ, *a*, una tantùm ſumptio vel poſitio eſt.

Duarum rerum, aut literarum, *a* & *b*, vel *a* præcedit & *b* ſequitur, vel præcedente *b* ſequitur *a*; unde duo ipſarum fiunt ordines *a b* & *b a*.

Tres, porrò, literæ *a*, *b*, *c*, ita collocari poſſunt, ut primus locus vel ipſi *a* vel *b* vel *c* concedatur: ſi *a* primum tenet locum, reliquæ duæ duobus, ut diximus, modis diſponi queunt: ſi *b* in primum locum transferatur, reliquarum duarum duplex itidem poterit eſſe poſitio; quod & intelligendum, ubi tertia *c* primam ſedem occupaverit. Unde trium literarum in univerſum ter duæ, ſeu 6, exiſtunt permutationes *a b c*, *a c b*: *b a c*, *b c a*: *c a b*, *c b a*.

Similitèr, ſi 4 extent literæ *a*, *b*, *c*, *d*, earum unaquæque primum obtinere locum poteſt, intereà dum tres reliquæ, ut nunc oſtenſum, ter bis, ſeu ſexiès, ordinem variabunt: quare cùm earum, quæ primo loco poni poſſunt, ſint quatuor, ſequitur omnes quatuor quater ter bis, ſeu quater ſexies, hoc eſt, vicies quater ſitum inter ſe permutare poſſe.

Ob eandem rationem accedente 5tâ literâ *e* inſtitui poſſunt quinquies tot variationes, quot in caſu præcedenti, hoc eſt, quinquies 24, ſeu 120. Et generalitèr, datis quotcunque literis, numerus permutationum, quas ſubire poſſunt omnes, totiès excedit numerum permutationum, quas recipiunt literæ unâ pauciores, quot ſunt unitates in dato literarum numero. Unde ſponte manat ſequens

Regula pro inveniendis omnibus permutationibus rerum quotcunque datarum.

OMNES numeri ab unitate ſe conſequentes naturali ordine, ad datum uſque rerum numerum incluſivè, ducantur in ſe invicèm ; productum manifeſtabit quæſitum.

Putà, ſi datus rerum numerus ſit n, numerus permutationum erit 1. 2. 3. 4. 5. &c. uſque ad n ; vel etiam (quia unitas non multiplicat) 2. 3. 4. 5. . . . . . n. Nota, punctula numeris interjecta hìc et ubique in ſimili materiâ continuum numerorum in ſe ductum ſignificant. Exempli gratiâ, ſeptem rerum permutationes ſunt 2. 3. 4. 5. 6. 7. = 5040. Ratio patet ex dictis, operatio ex adjunctâ Tabellâ ;

Numerus Rerum.	*Numerus Permutationum.*
1	1
	2
2	2
	3
3	6
	4
4	24
	5
5	120
	6
6	720
	7
7	5040
	8
8	40320
	9
9	362,880
	10
10	3,628,800
	3628800
11	39,916,800
	79833600
12	479,001,600

2. *Si rerum permutandarum nonnullæ ſunt eædem:*

Quòd ſi literæ una pluréſve recurrant ſæpiùs, hoc eſt, ſi in dato rerum numero aliquæ res ſimiles ſint ſive eædem; ut, ſi datæ ſint *a a a b c d*, ubi litera *a* ter repetitur; numerus permutationum multo minor evadit: ad quem inveniendum cogitandum eſt, quòd, ſi omnes eſſent diverſæ, putâ, ſi loco *a a a* ſcriberetur *a α* a, poſſent hæ tres literæ etiam nullâ cæterarum loco motâ inter ſe ſexiès tranſponi, per præcedentem Regulam; unde totidem diverſæ naſcerentur permutationes; at nunc cùm ſunt eædem, ſex iſtæ permutationes literarum *a α* a nullam univerſarum diſpoſitioni variationem indicunt, ac proinde pro unâ eâdemque habendæ ſunt: quod cùm de quâcunque diſpoſitione literarum paritèr ſit intelligendum, indicium præbet, numerum permutationum rerum datarum ſexiès, hoc eſt, totiès minorem eſſe numero permutationum, quas ſubire poſſent ſi omnes eſſent diverſæ, quotiès inter ſe permutari queunt res ſimiles: ſed ſi omnes 6 literæ diverſæ exiſterent, permutari poſſent, juxtà præcedentem tabellam, 720. vicibus. Ergò nunc ubi tres ipſarum conveniunt, permutari duntaxat poterunt vicibus 120.

Iterùm ſi datæ ſint 6 literæ *a a a b b c*, ubi præter literam *a* quæ ter recurrit, etiam litera *b* bis repetitur; manifeſtum eſt, numerum permutationum adhuc bis minorem evadere, quàm in præcedenti caſu fuerat, adeóque

que ſolùm ad 60 ſe extendere : quandoquidem binæ quælibet permutationes, quæ ex ſolâ tranſpoſitione duplici literarum *b b*, ſi diverſæ eſſent, naſcerentur, nunc coïncidunt. Eodem pacto colligendum, ſi plures literæ repeterentur ſæpiùs, pro ſingulis earum numerum permutationum minui toties, quoties ſeorsìm inter ſe permutari poſſunt eædem literæ. Unde ratio habetur ſequentis Regulæ.

Regula pro inveniendis rerum permutationibus, cùm earum nonnullæ ſunt eædem.

NUMERUS permutationum, quas admitterent datæ res ſi omnes differentes eſſent, dividatur per numerum permutationum, quas ſubire poteſt res ſimilis ſecundùm multitudinem ſuam, ſi una ſit quæ ſæpiùs repetatur : aut per productum ex numeris permutationum, quas ſeorsìm recipere poſſunt ſingulæ res ſimiles ſecundùm multitudinem ſuam, ſi plures ſint quæ ſæpiùs recurrant ; & quotiens exhibebit quæſitum.

Uſus doctrinæ Permutationum inſignis eſt in definiendo numero Anagrammatum alicujus vocis. Exempli gratiâ ; Tranſpoſitiones omnes poſſibiles literarum in voce *Roma* ſunt 1.2.3.4. = 24, ob 4 differentes literas, per 1 Regulam : et in voce *Leopoldus* $\frac{362880}{2.2=4} = 90720$: et in voce *Studioſus* $\frac{362880}{2.6=12} =$ 30240 : ob 9 utrobique literas, intérque illas ibi geminum *l* et geminum *o*, hìc geminum *u* & triplex *s* ; per 2 Regulam.

Hùc pertinent verſus nonnulli ob variationum multitudinem *Protei* dicti, quos inter celebrantur Lanſii, Scaligeri, Bauhuſii. Thomæ Lanſio hoc diſtichon debemus :

Lex, *Rex*, *Grex*, *Res*, *Spes*, *Jus*, *Thus*, *Sal*, *Sol*, (bona)
Lux, *Laus* :
Mars, *Mens*, *Sors*, *Lis*, *Vis*, *Styx*, *Pus*, *Nox*, *Fæx*, (mala)
Crux, *Fraus*.

[illegible]us ſinguli verſus per Regulam primam, ob 11 monoſyl-[illegible] (diſſyllabis vocibus *bona & mala* 5tæ ſemper regioni affixis)

fixis) ſalvâ metri lege variari poſſunt 39,916,800 vicibus. Et quanquam aliàs contingat, ut pleræque variationes in metri leges arietent, nec non ut plerique Anagrammatiſmi ſint non-ſignificantes & barbari; levi tamen plerumque induſtriâ opus eſt ad ſecernendum utiles ab inutilibus, illorúmque numerum ſeorsìm ineundum, ſi aliquem in iis inquirendis ordinem obſerves. Quemadmodùm cernere eſt in hexametro à Bernhardo Bauhuſio, Jeſuitâ Lovanienſi, in laudem Virginis Deiparæ conſtructo:

Tot tibi ſunt dotes, Virgo, quot ſidera cœlo;

quem dignum peculiari operâ duxerunt plures viri celebres. Erycius Puteanus in libello, quem Thaumata Pietatis inſcripſit, variationes ejus utiles integris 48 paginis enumerat, eâſque numero ſtellarum, quarum vulgò 1022 recenſentur, accommodat, omiſſis ſcrupuloſiùs illis, quæ dicere videntur, tot ſidera cœlo eſſe, quot Mariæ dotes; nam Mariæ dotes eſſe multo plures. Eundem numerum 1022 ex Puteano repetit Gerhardus Voſſius cap. 7. de Scientiis Mathematicis; Preſtetus, Gallus, in primâ editione Elementorum Mathematicorum, paginâ 348, Proteo huic 2196 variationes attribuit; ſed, factâ reviſione in alterâ editione, tomo primo, paginâ 133, numerum earum dimidio ferè auctum ad 3276 extendit. Induſtrii Auctorum Lipſienſium Collectores menſe Junii 1686, in recenſione Tractatûs Walliſiani de Algebrâ, numerum in quæſtione (quem Auctor ipſe definire non fuit auſus) ad 2580 determinant. Et ipſe poſtmodùm Walliſius in editione Latinâ operis ſui, Oxoniæ anno 1693 impreſſâ, paginâ 494, eundem ad 3096 profert. Sed omnes adhuc à vero ſunt deficientes, adeò ut deluſam tot virorum, poſt adhibitas quoque ſecundas curas, in re levi perſpicaciam meritò mireris. Facto enim examine deprehendo, fœtum hunc Bauhuſianum, excluſis etiàm ſpondaïcis, admiſſis verò iis qui cæſurâ deſtituti ſunt, ſalvâ metri lege omninò tèr milliès, tercentiès, ac duodeciès variabilem eſſe. At prolixiùs de his agere tanti non intereſt, nec inſtitutum noſtrum patitur.

TYPUS

TYPUS VARIATIONUM VERSÛS BAUHUSIANI:

Tot tibi sunt dotes, Virgo, quot sidera cælo.

Quintam Regionem Hexametri occupat vel

Sidera, quam vocem excipit aut vox

- *Dissyllaba una*, nempe vel
  - *Cælo*, ac tum vox *Tibi* inter sex reliquas occupat locum vel
    - *Secundum*, præcedente voce nunc
      - *Monosyllabâ*, eâque vel
        - *Tot*, cui casui respondent - Variationes 24
        - *Sunt*, - - - 24
        - *Quot*, - - 24
      - *Dissyllabâ Virgo*, - - 24
    - *Tertium*, præeuntibus
      - *Unâ monosyllabâ & unâ dissyllabâ*, primas tenente vel
        - *Monosyllabâ*, *Tot*, quam excipit alterutra
          - *Dotes*: 6, *Virgo*: 6 — 12
          - *Sunt*, - 12
          - *Quot*, - - - 12
        - *Dissyllabâ*, *Dotes*, quam sequitur
          - *Tot*: 6, *Sunt*: 6, *Quot*: 6 - - 18
          - *Virgo*, - - - 18
      - *Duabus dissyllabis*, nempe, *Dotes Virgo*, 6
    - *Quartum*, præcedentibus
      - *Tribus monosyllabis*, - - 12

Duabus

| | *Duabus monosyllabis cum dissyllabâ Virgo,* 12
| | *Una monosyllabâ & duabus dissyllabis,* 36
| *Quintum,* præmissis
| | *Tribus monosyllabis cum unâ dissyllabâ,* 48
| | *Duabus monosyllabis cum totidem dissyllabis,* quarum posterior *Virgo,* - 18
| | *Sextum,* - - - 120
a b c d -

420..420

| *Dotes,* unde totidem variationes, quot in *Cælo,* nempe - - - 420
| *Virgo,* unde rursùs totidem, quot in *Cælo,* exceptis solùm illis 60 variationibus, ubi postrema syllaba in *Virgo* correpta est; quibus proin demptis ex 420, remanent - - 360
| *Monosyllabæ duæ,* eáeque
Quot sunt, vel *Sunt quot*; voce *Tibi* occupante locum vel
| *Secundum,* primo relicto voci
| | *Monosyllabæ, Tot:* - - 12
| *Dissyllabæ, Virgo:* - - 12
| | *Tertium,* præcedentibus
Monosyllabâ cum dissyllabâ, - 24
| | *Duabus dissyllabis,* quarum post *Virgo,* 8
Quartum, præeuntibus
| | *Monosyllabâ cum duabus dissyllabis,* 36
Tribus dissyllabis, quarum ultima *Virgo,* 4
| | *Quintum,* - - - 48

| | 144..144

| *Tot sunt,* vel, *Sunt tot,* totidem - 144
Tot quot, aut, *Quot tot,* totidem - - 144

Tibi, quam vocem sequitur vox 1632
Dissyllaba una, eáque vel
| *Cælo,* voce *Sidera* occupante locum aut

Primum, - - -	120	
Secundum, - - - -	48	
Tertium, præmissis vel		
Duabus monosyllabis, - -	36	
Duabus dissyllabis, - -	12	
Quartum, præeuntibus duabus monosyllabis & unâ dissyllabâ, -	72	
Quintum, præcedentibus duabus monosyllabis, totidémque dissyllabis, -	72	
	360..	360
Dotes, totidem quot in *Cœlo* -		360
Virgo, totidem - - -		360
Monosyllabæ duæ, eáeque		
Quot sunt, vel *Sunt quot*: voce *Sidera* tenente locum		
Primum: - -	48	
Secundum, post dissyllabam vocem, -	36	
Tertium, post duas dissyllabas, -	24	
Quartum, post tres dissyllabas, -	12	
	120	.120
Tot sunt, vel, *Sunt tot*, totidem -		120
Tot quot, vel, *Quot tot*, totidem - -		120
Monosyllabâ unâ (quo casu ante *Tibi* semper habetur *Virgŏ*), nempe vel		
Sunt, voce *Sidera* locum possidente aut		
Primum: - - -	24	
Secundum: - -	12	
Tertium, præcedentibus *duabus monosyllabis*,	4	
duabus dissyllabis,	4	
Quartum, - - -	12	
Quintum, - - -	24	
	80...	80
Tot, totidem quot in *Sunt*, - -		80
Quot, totidem - - -		80

Summa omnium Variationum utilium 3312

C A P.

C A P. II.

DE COMBINATIONIBUS, IISQUE PRIMÒ CONSIDERATIS SIMPLICITÈR.

COMBINATIONES rerum ſunt conjunctiones, juxtà quas ex datâ rerum multitudine nonnullæ eximuntur, intérque ſe conjunguntur nullo ordinis ſitûs-ve ipſarum reſpectu habito.

Idcircò cùm quæritur, quoties ex dato rerum numero vel binæ, vel ternæ, vel quaternæ, &c. accipi poſſint, ſic ut nunquam omnes eædem res ſumantur ſæpiùs quàm ſemèl, dicentur quæri omnes combinationes diverſæ rerum datarum.

Numerus, ſecundùm quem res datæ conjunguntur, dicitur *Exponens* Combinationis: Ita, ſi res binæ ſumuntur, Exponens erit 2; ſi ternæ, 3; ſi quaternæ, 4. Res verò ſecundùm hos exponentes junctæ dicuntur *Binarii*, *Ternarii*, *Quaternarii*, &c. vel *Biniones*, *Terniones*, *Quaterniones*, &c. & conſonantèr etiàm *Uniones*, vel *Unitates*, quando res ſumuntur ſingulæ, & *Nulliones* cùm nulla planè ſumitur.

Conjunctiones ipſas nonnulli vocant *Combinationes*, *Conternationes*, *Conquaternationes*, &c. quas omnes vulgò unâ voce *Combinationum* complecti ſolent, tametſi hæc vox ſtrictiori ſignificatu propriè non niſi illas conjunctiones indigitare videatur, quibus res binæ invicèm junguntur. Quamobrem alii generaliori voce *Complicationum* vel *Complexionum* uti malunt: alii magìs appoſitè *Electiones* vocant, ut & illæ ſubintelligi poſſint rerum acceptiones, quibus res ſingulæ ſeorsìm ſumuntur, aut quibus etiàm nulla planè ſumitur.

Res autèm quæ inter ſe combinandæ ſunt, vel omnes poſſunt eſſe diverſæ, vel aliquot ipſarum eædem; eáeque vel ita combinari debent, ut in nullâ combinatione res eadem ſæpiùs contineatur, quàm ipſa reperitur in toto rerum numero: vel ſic, ut in eâdem combinatione res eadem etiàm ſæpiùs recurrere, hoc eſt, ut ſecum ipſâ quoque combinari poſſit. Iterúmque quæri poteſt numerus combinationum vel ſecundùm omnes exponentes conjunctìm, vel ſecundùm ſingulos ſeorsìm. Atque inſupèr circà unumquemque horum combinandi modorum plures formari poſſunt quæſtiones & problemata, è quibus illa tantùm delibabimus, quæ in ſequentibus alicui uſui fore judicamus.

1. *Si res omnes combinandæ ſunt diverſæ, ínque nullâ combinatione eadem res bìs occurrere debet, invenire omnes Combinationes ſimplicitèr ſive ſecundùm omnes exponentes conjunctìm.*

SUNTO combinandæ modis omnibus literæ *a*, *b*, *c*, *d*, *e*, &c. Fiant tot ſeries quot literæ, hoc modo: In primâ ſerie ponatur ſola litera *a*.

In ſecundâ ponatur *b*, nunc ſeorsìm, nunc junctìm cum *a*, ut habeatur *ab* vel *ba*. Eadem enim conjunctio eſt, quæ *b* cum *a*, & *a* cum *b* jungit, cùm nullus ordinis, ſitûs-ve ipſarum inter ſe, reſpectus haberi ſupponatur.

In tertiâ collocetur *c*, eáque primò ſola, dein juncta, partìm cum *a* & *b*, ut fiant biniones *ac*, *bc*; partìm cum ipſo binione *ab*, ut fiat ternio *abc*.

a.

b. ab.

c. ac. bc. abc.

d. ad. bd. cd. abd. acd. bcd. abcd.

e. ae. be. ce. de. abe. ace. bce. ade. bde. cde. abce. abde. acde. bcde. abcde.

In quartâ ponatur *d*, primò ſola, deinde juncta cum ſingulis præcedentium literarum *a*, *b*, *c*, ſingulíſque earum tum binariis *ab*, *ac*, *bc*, tum ternario *abc*; ut fiant novi biniones *ad*, *bd*, *cd*, terniones *abd*, *acd*, *bcd*, & quaternio *abcd*.

Similitèr quintæ ſeriei agmen ducat litera *e*, quam primò ingrediatur ſola, dein juncta cum omnibus præcedentium ſerierum electionibus. Eâdémque methodo procedendum eſſet,

ſi

ſi plures eſſent datæ literæ. Quâ ratione ſatis manifeſtum eſt, datas literas in iſtis ſeriebus omnifariàm inter ſe junctas eſſe, nullámque earum fieri poſſe electionem, quæ non in unâ harum ſerierum reperiatur, ſed & nullam eſſe quæ alicubi bìs occurrat; adeóque omnes unà ſeries ſuppeditaturas omnes electiones poſſibiles, quæ circà datas literas inſtitui queunt.

Harum igitur numerus initur facilè, ſi conſideretur quòd in quâlibet ſemper ſerie una ampliùs inveniri debeat electio, quàm in antecedentibus omnibus ſeriebus ſimul: quoniam litera, quæ illius ſeriei caput eſt, ibidem ſemel ponitur ſola, & prætereà unà aſſumit ſecum omnes electiones præcedentium ſerierum. Hinc enim ſequitur, quia in primâ ſerie eſt electio unica, fore in ſecundâ electiones duas, in tertiâ 4, in quartâ 8, & ſic deinceps in progreſſione geometricâ duplâ: quandoquidem progreſſionis duplæ ab unitate hanc quoque naturam eſſe conſtat *, ut ſumma terminorum quotlibet unitate aucta ſequentem terminum exhibeat. Quocircà ſumma electionum in ſeriebus omnibus æqualis eſt ſummæ terminorum totidem progreſſionis duplæ ab unitate, hoc eſt, per modò memoratam proprietatem, ipſi termino ſubſequenti

* Hoc autèm ita demonſtrari poteſt.

PROPOSITIO.

Sit ſeries terminorum in geometricâ ratione unitatis ad numerum binarium continuò creſcentium, ſcilicet, 1, 2, 4, 16, 32, 64, 128, 256, &c, uſque ad n terminos. Horum terminorum ſumma vocetur S. Manifeſtum eſt ultimum, ſive maximum, hujus ſeriei terminum fore æqualem $\overline{2}|^{n-1}$. Augeatur jam hæc ſeries uno adjecto termino, ſcilicet, $2 \times \overline{2}|^{n-1}$, ſeu $\overline{2}|^{n}$. Dico, quòd novus terminus $\overline{2}|^{n}$ erit æqualis $S + 1$, ſive ſummæ S omnium priorum terminorum unà cum unitate.

DEMONSTRATIO.

Duplicando terminos ſeriei S, ſive $1 + 2 + 4 + 8 + 16 + 32 + 64 +$ & $+ 2^{n-1}$, orietur ſeries $2 + 4 + 8 + 16 + 32 + 64 + 128 +$ &c. $+ \overline{2}|^{n}$, cujus termini omnes, excepto ultimo $\overline{2}|^{n}$, ſunt reſpectivè æquales terminis omnibus prioris ſeriei, excepto primo 1; hoc eſt, $2S$ erit $= S - 1 + \overline{2}|^{n}$. Ergò $2S + 1$ erit $= S + \overline{2}|^{n}$, et $\overline{2}|^{n}$ erit $= 2S + 1 - S$, ſeu $S + 1$.

Q. E. D.

ejuſdem

ejuſdem progreſſionis unitate multato; qui quidem terminus ſubſequens idem eſt cum producto binarii toties, ſive tot vicibus, poſiti & in ſe ducti, quot ipſum in progreſſione termini præcedunt, hoc eſt, quot ſunt ſeries, quarum electiones quæruntur. Unde talis exurgit.

Regula pro inveniendis omnibus electionibus rerum datarum ſecundùm omnes exponentes:

A Producto binarii toties, ſive tot vicibus, poſiti & multiplicati in ſe, quot ſunt datæ res, auferatur unitas: reliquum indicabit quæſitum.

Hoc eſt, poſito rerum datarum numero n, numerus omnium electionum ſimplicitèr, putà, omnium unionum, binionum, ternionum, &c. erit $2^n - 1$. Hinc ſi nullionem ſeu electionem, quâ ex rebus datis nulla ſumitur, quæque in quâvis rerum multitudine una ſemper eſt & unica, ſimùl comprehendas, fiet numerus ille 2^n: ſin cum nullione ipſos quoque uniones reſeces, quorum numerus ipſi rerum numero perpetuò æquatur, erit numerus binionum, ternionum, cæterarúmque complexionum $2^n - n - 1$. Exempli gratiâ. Septèm planetarum conjunctiones, vel complicationes, omnes diverſæ ſunt $2^7 - 1 = 2.2.2.2.2.2.2 - 1 = 128 - 1 = 127$; unde ſi demas electiones 7, quibus ſinguli planetæ ſeorſìm accipiuntur, quáeque propriè non conjunctiones, ſed disjunctiones planetarum ſunt, relinquetur numerus omnium conjunctionum ſtrictè dictarum, quibus planetæ vel bini, vel terni, vel quaterni, vel quini, vel ſeni, vel denique ſepteni junguntur, $2^7 - 7 - 1 = 120$. Sic etiàm duodecìm, uti vocant, Regiſtra, ſeu fiſtularum ordines, in organo pneumatico, quibus ſonus, mox ſibilans, mox tremebundus, efficitur, aut alitèr modificatur, variari poſſunt $2^{12} - 1 = 4095$ vicibus.

Nota: Si quis examinet ſeries combinationum ſuprà in typo expoſitas, obſervabit in qualibet ſerie (ſolâ primâ exceptâ,

ceptâ, quæ unicum unionem *a* complectitur) numerum electionum secundùm exponentes pares æquari numero electionum secundùm impares: saltèm, cùm id in aliquot ab initio seriebus verum deprehenderit, idem quoque in serie proximè sequente locum habere concludet. Nam litera, quæ illius seriei caput est, juncta præcedentium serierum electionibus iis, quæ impares exponentes habent, parium; & iis vicissìm quæ pares habent juncta, imparium; exponentium complexiones efficit: adscifcens verò primæ seriei unionem *a*, paris; & ipsa per se sola accepta, imparis; exponentis electionem constituit: unde & in hâc serie numerum harum numero illarum æquari constat. In omnibus igitur seriebus simùl sumtis numerus electionum secundùm impares exponentes numerum electionum secundùm pares unitate superabit; aut, si his insupèr nullionem accenseas, æquabit. Quocircà, cùm numerus omnium electionum simplicitèr, incluso nullione, ostensus sit 2^n, erit ejus semissis, sive potestas binarii proximè minor, 2^{n-1}, numerus electionum secundùm solos impares; &, dempto rursùm nullione, $2^{n-1} - 1$ numerus electionum secundùm solos pares exponentes. Idem quoque demonstrabitur infrà in coroll. 6. cap. 4.

CAP.

C A P. III.

DE COMBINATIONIBUS SECUNDÙM SINGULOS EXPONENTES SEORSÌM; UBI DE NUMERIS FIGURATIS, EORÚMQUE PROPRIETATIBUS AGITUR.

Ex typo combinationum præcedentis capitis manifeſtum fit, literam quæ cujuſlibet ſeriei caput eſt, adjunctam unionibus ſerierum præcedentium efficere ſuæ ſeriei biniones, adjunctam binionibus efficere terniones, ternionibus quaterniones, & ſic porrò: adeóque numerum binionum in quâvis ſerie æquari ſummæ unionum in omnibus ſeriebus antecedentibus, numerum ternionum ſummæ binionum, numerum quaternionum ſummæ ternionum, & generalitèr numerum combinationum ſecundùm datum quemcunque exponentem in ſerie quâcunque æquari ſummæ combinationum omnium præcedentium ſerierum ſecundùm exponentem unitate minorem dato. Sequitur hinc, quòd

Uniones, quia in ſingulis ſeriebus reperiuntur ſinguli, omnes inter ſe conſtituunt ſeriem 1.1.1.1.1. &c. ſeu ſeriem unitatum.

Biniones in primâ ſerie nulli ſunt, in ſecundâ 1, in tertiâ $1+1=2$, in 4tâ $1+1+1=3$, in 5tâ $1+1+1+1=4$, &c. proinde omnes biniones inter ſe conſtituunt ſeriem 0. 1. 2. 3. 4. 5. &c, hoc eſt, ſeriem *numerorum* arithmeticè progreſſionalium, ſive *Lateralium*.

Terniones in primâ & ſecundâ ſerie nulli ſunt, in 3tiâ 1, in 4tâ $1+2=3$, in 5tâ $1+2+3=6$, in 6tâ $1+2+3+4=10$. &c. Omnes itaque ordine accepti ſeriem conficiunt 0. 0. 1. 3. 6. 10. 15. &c. hoc eſt, ſeriem *numerorum*, ut vocant, *Trigonalium*, ſeu *Triangularium*.

Quaterniones in tribus primis ſeriebus nulli ſunt, in 4tâ 1, in 5tâ $1+3=4$, in 6tâ $1+3+6=10$, in 7mâ $1+3+6+10=20$. &c. qui omnes ordine aſſumti ſeriem efficiunt 0. 0 0. 1. 4. 10. 20. &c. ſeriem, videlicet, *Pyramidalium*.

Pari

Pari ratione Quiniones omnes feriem conftituunt *Trianguli-pyramidalium* 0. 0. 0. 0. 1. 5. 15. 35. &c. Seniones feriem *Pyramido-pyramidalium* 0. 0. 0. 0. 0. 1. 6. 21. &c. aliáeque combinationes fecundùm altiores exponentes efficiunt alias atque alias feries numerorum figuratorum altioris generis in infinitum.*

Et fic occafione doctrinæ Combinationum in fpeculationem infperatam *numerorum figuratorum* incidimus; quâ appellatione vulgò infigniuntur numeri, qui ex continuâ arithmeticè proportionalium, indéque ortorum numerorum, additione, vel collectione, generantur.

Ut verò hæ figuratorum numerorum feries fub unum afpectum caderent, eóque faciliùs comprehenderentur quæ de illis dicenda fuperfunt, fequentem appofui tabellam, quam quis nullo negotio quoufque voluerit tum deorfùm tum

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* De horum numerorum nominibus eft inter auctores arithmeticos quædam variatio. Nam numeri 0.0.0.0.1.5.15.35, &c, qui hìc vocantur *Trianguli-pyramidales*, vocantur à quibufdam fcriptoribus, et, inter alios, à Nicolao Mercatore, in celeberrimâ fuâ Logarithmotechniâ, *Trigono-trigonales*; et numeri 0.0.0.0.0.1.6.21.56, &c. qui hìc vocantur *Pyramido-pyramidales*, ab illo vocantur *Trigono-pyramidales*. Nomina quibus diverfi ordines numerorum figuratorum defignantur apud *Mercatorem* funt quæ fequuntur; fcilicet, 1. unitates, 2. radices, 3. numeri trigonales, 4. pyramidales, 5. trigono-trigonales, 6. trigono-pyramidales, 7. pyramidi-pyramidales, 8. trigono-trigono-pyramidales, 9. trigono-pyramidi-pyramidales, 10. pyramidi-pyramidi-pyramidales. Vide Scriptores Logarithmicos, tom. 1mum, pag. 178.—— Ad evitandam hanc confufionem nominum fatius effe videtur diverfos horum numerorum ordines poft quartum ordinem, five numerorum pyramidalium, diftinguere folùm per numeros exponentes ordinum defignandorum, appellando eos five numeros figuratos ordinis quinti, five ordinis fexti, five ordinis feptimi, five octavi, five noni, five decimi, aut alius cujufcunque ordinis.

dextrorſùm continuabit. Numeri barbari, ſeu Arabici, in ſiniſtro tabulæ margine adſcripti numerant columnas tranſverſas, & ſimùl rerum combinandarum multitudinem: nu-

TABULA COMBINATIONUM, SEU NUMERORUM FIGURATORUM.

EXPONENTES COMBINATIONUM.

Numeri Rerum Combinandarum.

	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
1.	1	0	0	0	0	0	0	0	0	0	0	0
2.	1	1	0	0	0	0	0	0	0	0	0	0
3.	1	2	1	0	0	0	0	0	0	0	0	0
4.	1	3	3	1	0	0	0	0	0	0	0	0
5.	1	4	6	4	1	0	0	0	0	0	0	0
6.	1	5	10	10	5	1	0	0	0	0	0	0
7.	1	6	15	20	15	6	1	0	0	0	0	0
8.	1	7	21	35	35	21	7	1	0	0	0	0
9.	1	8	28	56	70	6	28	8	1	0	0	0
10.	1	9	36	84	126	126	84	36	9	1	0	0
11.	1	10	45	120	210	252	210	120	45	10	1	0
12.	1	11	55	165	330	462	462	330	165	55	11	1

meri verò Romani in ſupremo margine conſpicui numerant columnas verticales & unà exponentes combinationum innuunt. Columnarum verticalium prima eſt ſeries monadum ſeu unitatum; ſecunda ſeries numerorum naturalium, ſeu lateralium, ab unâ cyphrâ incipiens; tertia ſeries trigonalium incipiens à cyphris duabus, quarta pyramidalium incipiens à tribus cyphris, quinta trianguli-pyramidalium incipiens à quatuor cyphris, & ſic deinceps,

Habet hæc tabula proprietates planè eximias & admirandas; præterquàm enim quòd Combinationum myſterium in illâ latere jam oſtendimus, notum eſt interioris geometriæ peritis, præcipua etiàm totius reliquæ matheſeos arcana inibi delitefcere. Nos proprietatum aliquas hìc delibabimus, & quidem delibabimus tantùm, nullius niſi primariæ illius, quæ propoſito noſtro inſervit, demonſtrationem accuratiorem allaturi, cùm cæteræ vel ex hâç oſtendi poſſint, vel ex ipſâ tabellæ

tabellæ conſtructione & numerorum figuratorum geneſi ſatis pateſcant.

Mirificæ Proprietates Tabulæ Combinationum:

1. Columnarum verticalium ſecunda incipit ab unâ cyphrâ, tertia à cyphris duabus, quarta à tribus: & generalitèr columna c à cyphris $c - 1$.

2. Columnarum verticalium termini primi ſignificativi à ſiniſtrâ dextrorſùm obliquè deſcendendo ordine ſumpti reddunt ipſos terminos primæ columnæ verticalis, ſecundi ſecundæ, tertii tertiæ, & ita deinceps: putà, primi conſtituunt ſeriem monadum, ſecundi lateralium, tertii trigonalium, &c.

3. Secundus ab unitate terminus columnæ verticalis cujuſlibet æquatur ipſius columnæ numero.

4. Terminus quivis tabellæ æquatur ſummæ omnium ſuperiorum præcedentis columnæ verticalis.

5. Quilibet terminus æquatur duobus aliis immediatè ſuprà ſe poſitis, quorum unus eſt in eâdem verticali columnâ, alter in præcedente.

6. Columnæ cujuſvis tranſverſæ termini ab unitate aliquouſque creſcunt, deinde per eoſdem gradus rursùm decreſcunt. Idem intellige de ſummis columnarum verticalium æque-altarum, ceu terminis ſequentis columnæ tranſverſæ per quartam proprietatem.

7. Columnarum verticalium æque-altarum baſes, ſive termini columnæ tranſverſæ cujuſlibet, primus quidem & ultimus ſignificativus perpetuò inter ſe æquantur, ut et ſecundus & penultimus, tertius & antepenultimus, atque ita porrò, ſi columna pluribus terminis ſignificativis conſtet.

8. Quin & ſumptis ab initio columnis verticalibus quotcunque cum totidem tranſverſis, collectiſque in unam ſummam qui in eâdem verticali ſibi reſpondent terminis, erit ſumma prima æqualis penultimæ, ſecunda antepenultimæ, tertia proantepenultimæ, & ſic deinceps. Exhibent enim hæ ſummæ ipſos columnæ tranſverſæ ſequentis terminos, primo excepto. Confer proprietates 4 & 7. Exempli gratiâ:

Quinque primæ columnæ tum verticales tum transversæ sunt:

1.	0.	0.	0.	0.
1.	1.	0.	0.	0.
1.	2.	1.	0.	0.
1.	3.	3.	1.	0.
1.	4.	6.	4.	1.
5.	10.	10.	5.	1.

Termini sextæ columnæ transversæ, primo excepto.

9. Columnæ transversæ ordine exhibent coëfficientes omnium potestatum à radice aliquâ binomiâ genitarum; nempe secunda coëfficientes radicis 1. 1. tertia quadrati 1. 2. 1. quarta cubi 1. 3. 3. 1. quinta biquadrati 1. 4. 6. 4. 1. & sic porrò.

10. Summæ serierum transversarum progrediuntur in continuâ ratione duplâ: summarum verò summæ ab initio collectæ terminos constituunt progressionis duplæ unitate multatos; putà

$$1 = 1$$
$$1+1 = 2$$
$$1+2+1 = 4$$
$$1+3+3+1 = 8$$
$$1+4+6+4+1 = 16$$

$$1 = 1 = 2 - 1$$
$$1+2 = 3 = 4 - 1$$
$$1+2+4 = 7 = 8 - 1$$
$$1+2+4+8 = 15 = 16 - 1$$
$$1+2+4+8+16 = 31 = 32 - 1$$

fluit ex iis quæ in præcedente capite de Combinationibus simpliciter spectatis dicta sunt.

11. Termini seriei verticalis cujuslibet ordine divisi per terminos collaterales seriei præcedentis (initio vel ab unitate vel à suis respectivè cyphris facto) exhibent quotos arithmeticè proportionales, quorum communis differentia est fractio, cujus numerator est unitas, & denominator ipse numerus, sive secundus

ſecundus ab unitate terminus ſeriei dividentis. Exempli gratiâ :

Diviſ.)	*divid.*	*(quot.*	*Diviſ.)*	*divid.*	*(quot.*
1)	1	(2 : 2	1)	0	(0 : 2
2)	3	(3 : 2	2)	1	(1 : 2
3)	6	(4 : 2	3)	2	(2 : 2
4)	10	(5 : 2	4)	6	(3 : 2
5)	15	(6 : 2	5)	10	(4 : 2

Diviſ.)	*divid.*	*(quot.*	*Diviſ.)*	*divid.*	*(quot.*
1)	1	(3 : 3	1)	0	(0 : 3
3)	2	(4 : 3	3)	1	(1 : 3
6)	10	(5 : 3	6)	4	(2 : 3
10)	20	(6 : 3	10)	10	(3 : 3
15)	35	(7 : 3	15)	20	(4 : 3

Non difficultèr hæc proprietas, ſi opus foret, deduci poſſet ex ſequente.

12. Summa terminorum quotcunque ſeriei verticalis cujuſlibet à ſuis reſpectivè cyphris incipientis ad ſummam terminorum totidem ultimo æqualium eam habet rationem, quam habet unitas ad illius ſeriei numerum; hoc eſt, aggregatum numerorum quotcunque lateralium ab unâ cyphrâ ſeriem auſpicantium eſt ad aggregatum numerorum totidem maximo eorum, ſeu ultimo, æqualium, ut 1 ad 2; trigonalium à cyphris duabus, ut 1 ad 3; pyramidalium à tribus, ut 1 ad 4; &c. Idem quoque valet de ratione, quam habet ſumma terminorum ſeriei cujuſlibet ab unitate incipientis ad ſummam totidem maximum ſequenti termino æqualium. Exempli gratiâ :

				0	6		
0	3	1	5	0	6	1	15
1	3	2	5	1	6	3	15
2	3	3	5	3	6	6	15
3	3	4	5	6	6	10	15
6. 12 :: 1. 2		10. 20 :: 1. 2		10. 30 :: 1. 3		20. 60 :: 1. 3	

0	10			
0	10	1	56	
0	10	4	56	
1	10	10	56	&c.
4	10	20	56	
10	10	35	56	
15.60 :: 1.4		70.280 :: 1.4		

Cùm inter affectiones numerorum figuratorum hæc præcipua sit, eadémque scopo nostro primario inserviat, visum hìc est exponere methodum, quâ talem proprietatis ἀπόδειξιν exhibeo, quæ simùl & scientifica sit, & propositum universalitèr concludat. Quem in finem sequentia præstruo lemmata:

Lemma Primum.

Summa terminorum quotlibet primæ seriei ad summam totidem terminorum ultimo æqualium rationem habet æqualitatis, sive ut 1 ad 1.

Demonstratio.

Cùm enim series meris constet unitatibus, erit summa terminorum quotlibet, summa tot unitatum, hoc est, tot terminorum ultimo æqualium, quot sunt termini.

Q. E. D.

Lemma Secundum.

In quâlibet serie à suis respectivè cyphris incipiente, si quota est ipsa inter series, tot ab initio sumantur termini, erit summa terminorum omnium ad summam totidem ultimo æqualium, ut 1 ad seriei numerum.

Demonstratio.

Numerus enim cyphrarum quamcunque seriem auspicantium unitate minor est seriei numero, per proprietatem primam. His igitùr ſi accedat ſequens terminus, numerus terminorum ſeriei numero æquabitur. Sed terminus, qui proximè cyphras ſequitur, eſt unitas, per proprietatem ſecundam. Unde terminorum aggregatum æquatur unitati, & aggregatum totidem ultimo æqualium æquatur ipſi ſeriei numero. Quarè conſtat Propoſitio.

Lemma Tertium.

In quâcunque numerorum ſerie, ſi ſumma terminorum ab initio ſumptorum ad ſummam totidem ultimo æqualium perpetuò eandem habeat rationem, quotcunque accipiantur termini, putà ut 1 ad R, ita ut ſumma terminorum æquetur ſummæ totidem ultimo æqualium diviſæ per R; erit numerus terminorum aſſumptorum ablato R ad eundem numerum unitate mulctatum, ut ſumptorum penultimus ad ultimum.

Demonstratio.

Sumpti ſint ab initio termini quotlibet A.B.C.D. quorum numerus ſit N, penultimus C, & ultimus D. Eſt utique $A + B + C = A + B + C + D - D$, hoc eſt, (per hypotheſin) $\frac{C \text{ in } \overline{N-1}}{R}$ eſt $= \frac{D \text{ in } N}{R} - D$, & proinde, æque-multiplicando, C in $\overline{N-1}$ erit $= D$ in $N - D$ in $R = D$ in $\overline{N-R}$, adeóque $N - R : N - 1 :: C : D$. Q. E. D.

Lemma Quartum.

In tabulà numerorum figuratorum ſi duæ ſint columnæ verticales contiguæ, in quarum priore quotlibet ab initio termini ad totidem ultimo eorum æquales habeant conſtantem rationem, ut 1 ad r; habeant verò in poſteriore termini aliquot ab initio ſumpti ad totidem ſumptorum ultimo æquales rationem ut 1 ad $r + 1$: habebit quoque, addito ſequenti termino,

termino, ſumma omnium terminorum unà cum adjecto ad tot terminos adjecto æquales, quot ſunt cum adjecto termini, rationem ut 1 ad $r+1$.

DEMONSTRATIO.

Sumpti ſint in poſteriore columnâ termini E.F.G.H, quos proximè ſequatur I; atque ſumantur in columnâ immediatè præcedente termini totidem A.B.C.D; ſumptorum verò utrinque numerus ſit n. Erit r H = (ex numerorum figuratorum geneſi per proprietatem quartam) r in A + B + C = (per hypotheſin) $\overline{n-1}$ in C = (per lemma tertium) $\overline{n-r}$ in D; quare $n-r : \text{H} :: r : \text{D} ::$ (per hypotheſin) $n : \text{A} + \text{B} + \text{C} + \text{D} ::$ (ex numeratorum figuratorum geneſi per proprietatem quartam) n. I. Unde $\overline{n-r}$ in I = n H = (per hypotheſin) $\overline{r+1}$ in E + F + G + H; adeoque $n-r : r+1 :: \text{E} + \text{F} + \text{G} + \text{H} : \text{I}$, &, componendo, $n+1 : r+1 :: \text{E} + \text{F} + \text{G} + \text{H} + \text{I} : \text{I}$, hoc eſt, $\text{E} + \text{F} + \text{G} + \text{H} + \text{I} : \overline{n+1}$ in $\text{I} :: 1 : r+1$ *. Q. E. D.

Cum

* Hæc demonſtratio præ nimiâ brevitate mihi videtur eſſe obſcura. Poteſt verò explicari et, ut opinor, ſatìs perſpicua reddi, modo ſequente.

Sumpti ſint in poſteriore columnâ termini E.F.G.H.; quos proximè ſequatur I; atque ſumantur in columnâ immediatè præcedente termini totidem A.B.C.D: ſumptorum verò utrinque numerus ſit n. Et ſit ſumma quotlibet terminorum A.B.C.D. ad totidem ultimo eorum æquales in ratione 1 ad r; et ſit ſumma terminorum E.F.G.H. ad n terminos ipſi H, eorum ultimo, æquales, hoc eſt, ad quantitatem $n \times$ H, in ratione 1 ad $r+1$. Dico, quòd ſumma omnium terminorum E. F. G. H. I erit ad $n+1$ terminos ipſi I æquales, hoc eſt, ad $\overline{n+1} \times$ I, ut 1 ad $r+1$.

DEMONSTRATIO.

Ex numerorum figuratorum geneſi, per proprietatem quartam ſuprà memoratam, erit $r \times$ H æqualis $r \times \overline{\text{A} + \text{B} + \text{C}}$, ideóque (per hypotheſin) æqualis $\overline{n-1} \times$ C, atque idcircò (per lemma tertium) æqualis $\overline{n-r} \times$ D. Erit igitùr $n-r$ ad H ut r ad D. Sed (per hypotheſin) A + B + C + D eſt ad $n \times$ D ut 1 ad r; et proinde (permutando) A + B + C + D erit ad 1 ut $n \times$ D ad r, et (invertendo) 1 erit ad A+B+C+D ut r ad n D.

Eſt

Cum olìm horum Fratri * copiam feciſſem, animadvertit ille poſſe demonſtrationem elegantèr abbreviari, poſtremis tribus lemmatibus in unum conflatis, hoc modo:

Lemma.

In tabulâ numerorum figuratorum ſi ſumma terminorum ab initio ſeriei verticalis cujuſvis ad ſummam totidem maximo æqualium ubique rationem habeat ut 1 ad r, habebit ſumma terminorum ſeriei proximè ſequentis ad ſummam totidem maximo æqualium rationem ut 1 ad $r + 1$.

Demonstratio.

Sint ſeries ſequentes $a.\ b.\ c.\ d.$ &c. & $o.\ g.\ h.\ i.$ &c. numerus terminorum prioris ſit n, poſterioris $n + 1$.

$$n\left\{\begin{array}{ccc} a & - - & o \\ b & - - & g \\ c & - - & h \\ d & - - & i \\ e & - - & l \\ f & - - & p \\ & & q \end{array}\right\} n + 1.$$

Eſt primò $q + p + l + i + h + g + o =$ (ex hypotheſi & geneſi numerorum figuratorum per proprietatem quartam) $\frac{nf}{r} + \frac{\overline{n-1}.e}{r} + \frac{\overline{n-2}.d}{r} + \frac{\overline{n-3}.e}{r} + \frac{\overline{n-4}.b}{r} + \frac{\overline{n-5}.a}{r} = \frac{n.\overline{f+e+d+c+b+a}}{r}$

Eſt autem $n \times 1$, ſeu n, ad 1, ut $n \times r$ eſt ad r. Ergò, ex æquo, $n \times 1$, ſeu n, erit ad $A+B+C+D$ ut $n \times r$ ad n D, hoc eſt, ut r ad D. Erit igitùr $n - r$ ad H ut n ad $A+B+C+D$.

Sed (ex numerorum figuratorum geneſi, per proprietatem quartam ſuprà memoratam) terminus I eſt æqualis $A + B + C + D$.

Erit igitùr $n-r$ ad H ut n ad I, et proinde $\overline{n-r} \times I$ erit $= n \times H$.

Sed, per hypotheſin, $E + F + G + H$ eſt ad $n \times H$ ut 1 ad $r + 1$; atque ideò $\overline{E+F+G+H} \times \overline{r+1}$ eſt æqualis $n \times H \times 1$, ſeu $n \times H$.

Erit igitùr $\overline{n-r} \times I = \overline{E+F+G+H} \times \overline{r+1}$; atque ideò erit $n-r$ ad $r+1$ ut $E+F+G+H$ ad I, et proinde (componendo) erit $n - r + r + 1$, ſeu $n + 1$, ad $r + 1$ ut $E + F + G + H + I$ ad I, et (permutando) $n + 1$ ad $E+F+G+H+I$ ut $r + 1$ ad I, et (invertendo) $E+F+G+H+I$ ad $n + 1$ ut I ad $r + 1$, et (multiplicando conſequentes per I) $E+F+G+H+I$ ad $\overline{n+1} \times I$ ut I ad $\overline{r+1} \times I$, hoc eſt, ut 1 ad $r+1$.

Q. E. D.

* Johanni Bernoullio.

$\frac{-e-2d-3c-4b-5a}{r}$ = (ex geneſi numerorum figuratorum) $\frac{nq-p-l-i-h-g}{r}$. Ergò $rq + r.\overline{p+l+i+h+g} = nq-p-l-i-h-g$; factâque tranſlatione convenienti, $\overline{r+1} \times \overline{p+l+i+h+g} = nq-rq$. Dividatur utrinque per $r+1$, erit $p+l+i+h+g = \frac{nq-rq}{r+1}$; additóque q habebitur $q+p+l+i+h+g = \frac{nq-rq}{r+1} + q = \frac{\overline{n+1}\times q}{r+1}$, hoc eſt, $g+h+i+l+p+q$ erit ad $\overline{n+1}\times q$ ut 1 ad $r+1$. Q. E. D.

Sequitur nunc *Propoſitio principalis*, quæ talis eſt.

Propositio Principalis.

In tabulâ numerorum figuratorum ſumma terminorum quotlibet à ſuis reſpectivè cyphris incipientium ad ſummam totidem ultimo æqualium: Item ſumma terminorum quotvis incipientium ab unitate ad ſummam totidem ultimum ſequenti æqualium: in ſerie primâ, ſeu monadum, eſt ut 1 ad 1; in ſerie ſecundâ, ſeu lateralium, ut 1 ad 2; in tertiâ, ſeu trigonalium, ut 1 ad 3; in quartâ, ſeu pyramidalium, ut 1 ad 4, & generalitèr in ſerie quâcunque ut 1 ad illius ſeriei numerum.

Demonstratio Primæ Partis hujusce Propositionis.

De primâ ſerie conſtat ex primo lemmate: de ſecundâ, tertiâ, quartâ, &c. è reliquis. Nam, quia ſumma terminorum quotlibet ad ſummam totidem ultimo æqualium in primâ ſerie eſt ut 1 ad 1, erit, vi horum lemmatum, in ſecundâ ut 1 ad $1+1=2$; &, quia in ſecundâ eſt ut 1 ad 2, erit in tertiâ ut 1 ad $2+1=3$; & propterea etiàm in quartâ ut 1 ad $3+1=4$; in quintâ ut 1 ad $4+1=5$; & generalitèr in ſerie c ut 1 ad c. Q. E. D.

Demon-

Demonstratio Secundæ Partis hujusce Propositionis.

Quia rationem 1 ad $r+1$ memoratam in ultimo lemmate hìc interpretamur per rationem 1 ad c, erit $r = c - 1 =$ (per proprietatem primam 1) numero cyphrarum, à quibus columna c incipit. Quarè, cum in dicto lemmate repertum sit $g+h+i+l+p = \frac{\overline{n-r}\times q}{r+1} = \frac{\overline{n-r}\times q}{c}$, sequitur quòd $g+h+i+l+p$ (summa terminorum quorum numerus est n) se habet ad q in $n-r$ (numerum terminorum minùs numero cyphrarum) sicut 1 ad c; hoc est, summa terminorum quotlibet ab unitate incipientium ad totidem terminos sequenti ultimum æquales, ut 1 ad c*. Q. E. D.

Consectarium.

Ex hâc ostensâ proprietate facile nunc est invenire tùm terminum optatum, tùm summam terminorum seriei cujuslibet. Sumpti intelligantur termini æque-multi ex pluribus continuè columnis, & sit numerus sumptorum ab initio cujusque columnæ n, adeóque numerus terminorum ab unitate (exclusis cyphris initialibus) in secundâ columnâ $n-1$, in tertiâ $n-2$, in quartâ $n-3$, atque ita deinceps, per primam proprietatem: quo posito, quæsitum ita colligo. Summa terminorum n primæ columnæ, nempe, n unitates, seu $\frac{n}{1}$, æquatur termino $\overline{n+1}$.no, hoc est, termino sequenti ultimum, secundæ columnæ, per quartam proprietatem, ex tabulæ genesi. Quarè termini hujus in $n-1$ (numerum terminorum ab unitate secundæ columnæ) ducti subduplum, seu $\frac{n.\overline{n-1}}{1.2}$, per duodecimam proprietatem æquale est aggregato terminorum secundæ columnæ, & simùl (per quartam proprietatem) ipsi termino sequenti ultimum tertiæ columnæ.

* Vide super hâc materiâ opera ipsius Johannis Bernoullii, edita Lausannæ anno Domini 1742, Tomum tertium, paginam 521, in 47mâ Lectione de Calculo Integralium.

Unde ſimilitèr hujus termini in $n-2$ (numerum terminorum ab unitate tertiæ columnæ) ducti ſubtriplum, nempe $\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}$, æquatur (per duodecimam proprietatem) aggregato terminorum tertiæ columnæ, inſimúlque (per quartam proprietatem) ipſi termino ſequenti ultimum quartæ. Quocircà & hujus termini in $n-3$ (numerum terminorum ab unitate quartæ columnæ) ducti ſubquadruplum, putà $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}$, exhibet ſummam terminorum quartæ columnæ, unáque terminum qui ſequitur ultimum quintæ; & rursùs iſtius termini in $n-4$ ducti ſubquintuplum nempe, $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3.4.5}$, producit ſummam terminorum columnæ quintæ, & ſimùl terminum qui excipit ultimum ſextæ; atque ita conſequentèr. E quibus igitùr infertur, quòd ſumma terminorum n primæ columnæ ſit $\frac{n}{1}$, ſecundæ $\frac{n.\overline{n-1}}{1.2}$, tertiæ $\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}$, quartæ $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}$, quintæ $\frac{n.\overline{n-1}\,\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3.4.5}$, & generalitèr columnæ c, $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}\ldots\overline{n-c+1}}{1.2.3.4.5\ldots c}$. Et, quia quælibet harum quantitatum etiàm exprimit terminum $n+1$ ſequentis columnæ, ſequitur quòd ipſe illius terminus optatus, ſeu ultimus, n habeatur mutato ſolummodò ubique n in $n-1$; adeóque quòd terminus optatus, ſecundæ columnæ ſit $\frac{n-1}{1}$, tertiæ $\frac{\overline{n-1}.\overline{n-2}}{1.2}$, quartæ $\frac{\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3}$, quintæ $\frac{\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.2.3.4}$, &, generalitèr, columnæ c, $\frac{\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}\ldots\overline{n-c+1}}{1.2.3.4\ldots c-1}$.

SCHOLIUM.

SCHOLIUM.

Multi, ut hoc in tranſitu notemus, numerorum figuratorum contemplationibus vacârunt (quos inter Faulhaberus & Remmelini Ulmenſes, Walliſius, Mercator in Logarithmotechniâ, Preſtetus, aliíque); ſed qui proprietatis hujus demonſtrationem univerſalem dederit & ſcientificam, novi neminem. Walliſius in Arithmeticâ Infinitorum fundamentum ſuæ methodi jacturus, rationes quas habent ſeries quadratorum, cuborum, aliarúmque poteſtatum, numerorum naturalium ad ſeriem totidem maximo æqualium, inductione inveſtigat; indéque in propoſitione 176, ad contemplationem numerorum trigonalium, pyramidalium, reliquorúmque figuratorum tranſit. Sed ſatius fuiſſet fortéque naturæ rei convenientius, ſi vice verſâ tractationem numerorum figuratorum, eámque univerſali & accuratâ demonſtratione munitam, præmiſiſſet, ac tum demùm ad poteſtatum ſummas inveſtigandas perrexiſſet. Præterquam enim quòd modus demonſtrandi per inductionem parùm ſcientificus eſt, inſupérque pro quâlibet ſerie peculiarem operam depoſcit; illa utique omnium judicio præcedere debent, quæ cæteris naturâ ſunt priora & ſimpliciora, quales videntur eſſe numeri figurati præ poteſtatibus, tùm quòd illi additione, hæ multiplicatione generantur, tùm, & præcipuè, quòd ſeries figuratorum à ſuis reſpectivè cyphris incipientes ad ſeries æqualium rationem habent exactè ſubmultiplicem, qualem non habere poſſunt ſeries poteſtatum (ſaltèm in terminis numero finitis) abſque aliquo exceſſu vel defectu, quicunque cyphrarum numerus ipſis præfigatur. De cætero namque ex cognitis figuratorum ſummis nihilo difficiliùs inveſtigari poterunt poteſtatum ſummæ, atque ex his priores collegit auctor; quod quomodo fiat, paucis oſtendam.

Inveſtigatio

Inveſtigatio Summarum quæ proveniunt ex additione quadratorum, cuborum, quadrato-quadratorum, et ſequentium poteſtatum numerorum naturalium 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. *ex numerorum figuratorum ſummis derivata.*

Proponatur ſeries numerorum naturalium ab unitate 1. 2. 3. 4. 5. &c. uſque ad n, & quærantur omnium ipſorum, item omnium quadratorum, cuborum, et ſequentium poteſtatum ex ipſis, ſummæ. Quoniam in tabulâ combinationum terminus ſecundæ columnæ indefinitè eſt $n-1$, & ſumma omnium terminorum, hoc eſt, ſumma omnium $n-1$, ſeu $\int.\overline{n-1}$, per confectarium præcedens inventa, eſt $\frac{n.\overline{n-1}}{1\,.\,2} = \frac{nn-n}{2}$, erit $\int.\overline{n-1}$, ſive $\int n - \int 1$, $= \frac{nn-n}{2}$, & proinde $\int n = \frac{nn-n}{2} + \int 1$; ſed $\int 1$ (ſumma omnium unitatum) eſt n; quarè ſumma omnium n, ſeu $\int n$, erit $= \frac{nn-n}{2} + n = \frac{1}{2}nn + \frac{1}{2}n$.

Porrò cum terminus, tertiæ columnæ indefinitè acceptus per idem confectarium ſit $\frac{n-1.\,n-2}{1\,.\,2} = \frac{nn-3n+2}{2}$, & ſumma omnium terminorum (hoc eſt), omnium $\frac{nn-3n+2}{2}$) $\frac{n.\,n-1.\,n-2}{1\,.\,2\,.\,3} = \frac{n^3-3nn+2n}{6}$; erit $\int\overline{\frac{nn-3n+2}{2}}$ ſive $\int\frac{1}{2}nn - \int\frac{3}{2}n + \int 1 = \frac{n^3-3nn+2n}{6}$, & $\int\frac{1}{2}nn = \frac{n^3-3nn+2n}{6} + \int\frac{3}{2}n - \int 1$; ſed $\int\frac{3}{2}n = \frac{3}{2}\int n =$ (per modò oſtenſa) $\frac{3}{4}nn + \frac{3}{4}n$, & $\int 1 = n$: unde his ſubſtitutis ſit $\int\frac{1}{2}nn = \frac{n^3-3nn+2n}{6} + \frac{3nn+3n}{4} - n = \frac{1}{6}n^3 + \frac{1}{4}nn + \frac{1}{12}n$, ejúſque duplum $\int nn$ (ſumma quadratorum ex omnibus n) $= \frac{1}{3}n^3 - \frac{1}{2}nn + \frac{1}{6}n$.

Rursùs,

Rursùs, quia terminus n quartæ columnæ eſt $\frac{n-1 . n-2 . n-3}{1 . 2 . 3} = \frac{n^3-6nn+11n-6}{6}$, & ſumma omnium terminorum $\frac{n . n-1 . n-2 . n-3}{1 . 2 . 3 . 4} = \frac{n^4-6n^3+11nn-6n}{24}$, erit utique $\int \frac{n^3-6nn+11n-6}{6}$, hoc eſt, $\int \frac{1}{6} n^3 - \int nn + \int \frac{11}{6} n - \int 1 = \frac{n^4-6n^3+11nn-6n}{24}$, indéque $\int \frac{1}{6} n^3 = \frac{n^4-6n^3+11nn-6n}{24} + \int nn - \int \frac{11}{6} n + \int 1$. Et quoniam per modò inventa $\int nn = \frac{1}{3} n^3 + \frac{1}{2} nn + \frac{1}{6} n$; nec non $\int \frac{11}{6} n$ ſive $\frac{11}{6} \int n = \frac{11}{12} nn + \frac{11}{12} n$, & $\int 1 = n$; hinc, factâ horum ſubſtitutione, emerget $\int \frac{1}{6} n^3 = \frac{n^4-6n^3+11nn-6n}{24} + \frac{1}{3} n^3 + \frac{1}{2} nn + \frac{1}{6} n - \frac{11}{12} nn - \frac{11}{12} n + n = \frac{1}{24} n^4 + \frac{1}{12} n^3 + \frac{1}{24} nn$, ejúſque proin ſextuplum $\int n^3$ (ſumma cuborum) $= \frac{1}{4} n^4 + \frac{1}{2} n^3 + \frac{1}{4} nn$. Atque ſic porrò ad altiores gradatìm poteſtates pergere, levíque negotio ſequentem adornare laterculum licet;

Summæ

Summæ Poteſtatum.

$$\int n = \tfrac{1}{2}nn + \tfrac{1}{2}n.$$
$$\int nn = \tfrac{1}{3}n^3 + \tfrac{1}{2}nn + \tfrac{1}{6}n.$$
$$\int n^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}nn.$$
$$\int n^4 = \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 \; * - \tfrac{1}{30}n.$$
$$\int n^5 = \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 \; * - \tfrac{1}{12}nn.$$
$$\int n^6 = \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 \; * - \tfrac{1}{6}n^3 \; * + \tfrac{1}{42}n.$$
$$\int n^7 = \tfrac{1}{8}n^8 + \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 \; * - \tfrac{7}{24}n^4 \; * + \tfrac{1}{12}nn.$$
$$\int n^8 = \tfrac{1}{9}n^9 + \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 \; * - \tfrac{7}{15}n^5 \; * + \tfrac{2}{9}n^3 \; * - \tfrac{1}{30}n.$$
$$\int n^9 = \tfrac{1}{10}n^{10} + \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 \; * - \tfrac{7}{10}n^6 \; * + \tfrac{1}{2}n^4 \; * - \tfrac{1}{12}nn.$$
$$\int n^{10} = \tfrac{1}{11}n^{11} + \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 \; * - 1n^7 \; * + 1n^5 \; * - \tfrac{1}{2}n^3 \; * + \tfrac{5}{66}n.$$

Quìn imò qui legem progreſſionis terminorum in hoc laterculo deſcriptorum attentiùs inſpexerit, eundem etiàm continuare poterit abſque his ratiociniorum ambagibus. Sumptâ enim *c* pro poteſtatis cujuſlibet exponente, fit ſumma ma omnium n^c, seu $\int n^c$, $= \frac{1}{c+1} n^{c+1} + \frac{1}{2} n^c +$

$$\frac{c}{2} A n^{c-1} + \frac{c\cdot\overline{c-1}\cdot\overline{c-2}}{2\cdot 3\cdot 4} B n^{c-3} + \frac{c\cdot\overline{c-1}\cdot\overline{c-2}\cdot\overline{c-3}\cdot\overline{c-4}}{2\cdot 3\cdot 4\cdot 5\cdot 6} C n^{c-5}$$

$$+ \frac{c\cdot\overline{c-1}\cdot\overline{c-2}\cdot\overline{c-3}\cdot\overline{c-4}\cdot\overline{c-5}\cdot\overline{c-6}}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} D n^{c-7} \; \ldots\ldots$$ & ita deinceps, exponentem poteſtatis ipſius n continuè minuendo binario, quousque perveniatur ad n vel nn. Literæ capitales A, B, C, D, &c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, &c. nempe $A = \frac{1}{6}$, $B = -\frac{1}{30}$, $C = \frac{1}{42}$, $D = -\frac{1}{30}$. Sunt autem hi coëfficientes ita comparati, ut ſinguli cum cæteris ſui ordinis coëfficientibus complere debeant unitatem; ſic D valere diximus $-\frac{1}{30}$, quia $\frac{1}{9} + \frac{1}{2} + \frac{2}{3} - \frac{7}{15} + \frac{2}{9}$ $(+D)$ $-\frac{1}{30} = 1$. Hujus laterculi beneficio intrà ſemi-quadrantem horæ reperi, quòd poteſtates decimæ, ſive quadrato-ſurſolidæ, mille primorum numerorum ab unitate in ſummam collecta efficiunt

91409924241424243424241924242500.

E quibus apparet, quàm inutilis cenſenda ſit opera Iſmaelis Bullialdi, quam conſcribendo tam ſpiſſo volumini Arithmeticæ ſuæ Infinitorum impendit, ubi nihil præſtitit aliud, quàm ut primarum tantum ſex poteſtatum ſummas (partem ejus quod unicâ nos conſecuti ſumus paginâ) immenſo labore demonſtratas exhiberet.

De ſeriebus ſerierum figuratarum analogis.

Antequam caput hoc finiamus, paucis adhuc indicare lubet quomodo, ſuppoſitis iis quæ de ſeriebus figuratis oſtenſa ſunt, poſſint quævis etiàm aliæ ſeries figuratarum analogæ (quæ, ſcilicet, differentias ſuas primas, ſecundas, tertias, &c. æquales habent, adeóque ex continuâ additione terminorum alicujus ſeriei æqualium generantur) ad homologas figuratas reduci, ac proinde ſummari, vel poſtremi ipſarum termini inveniri. Sit ſeries quævis æqualium D, ex cujus additione naſcatur ſeries C, & ex hujus additione ſeries B, & ex hujus tandèm collectione ſeries A, ſumptis ad arbitrium primis ſerierum terminis *d*, *c*, *b*, *a*. Vocabitur ſeries A figuratarum analoga, cujus differentiæ primæ conſtituunt ſeriem B, ſecundæ ſeriem C, tertiæ ſeriem D, &c. Et quoniàm appa-

D	C	B	A.
d	c	b	a
d	$c+d$	$b+c$	$a+b$
d	$c+2d$	$b+2c+d$	$a+2b+c$
d	$c+3d$	$b+3c+3d$	$a+3b+3c+d$
d	$c+4d$	$b+4c+6d$	$a+4b+6c+4d$
d	$c+5d$	$b+5c+10d$	$a+5b+10c+10d$

ret, ſeriem A componi ex ſeriebus unitatum 1, 1, 1, 1, &c. lateralium 1, 2, 3, 4, &c. trigonalium 1, 3, 6, 10, &c. pyramidalium 1, 4, 10, 20, &c. in primos differentiarum terminos *a*, *b*, *c*, *d*, ſeorsìm ductis, quarúmque omnium poſtremi termini & ſummæ per ante dicta habentur, ipſius quoque hinc ſeriei A poſtremum terminum & ſummam terminorum obtineri poſſe conſtat; nimirùm, ſi numerus terminorum vocetur n, erit ultimus terminus ſeriei A $= a + \overline{n-1}.\, b + \frac{\overline{n-1}.\overline{n-2}}{2}\, c + \frac{\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3}\, d$; & ſumma omnium terminorum $= na + \frac{n.\overline{n-1}}{2}\, b + \frac{n.\overline{n-1}.\overline{n-2}}{2.3}\, c + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{2.3.4}\, d$.

A

TRANSLATION

OF THE

THREE FIRST CHAPTERS

OF THE

SECOND PART, OR BOCK,

OF

MR. JAMES BERNOULLI's EXCELLENT TREATISE,

INTITLED

ARS CONJECTANDI;

OR

" THE ART OF FORMING PROBABLE CONJECTURES CONCERNING EVENTS THAT DEPEND ON CHANCE."

Publiſhed in a ſmall Quarto Volume at BASIL, or BASLE, in Switzerland, in the Year 1713.

THE PROŒMIUM, OR PREFACE, TO THE SECOND PART OF THE SAID TREATISE

DE ARTE CONJECTANDI.

IT is eaſy to perceive that the prodigious variety which appears both in the works of nature and in the actions of men, and which conſtitutes the greateſt part of the beauty of the univerſe, is owing to the multitude of different ways in which its ſeveral parts are mixed with, or placed near, each other. But, becauſe the number of cauſes that concur in producing a given event, or effect, is oftentimes ſo immenſely great, and the cauſes themſelves are ſo different one from another, that it is extremely difficult to reckon up all the different ways in which they may be arranged, or combined together, it often happens that men, even of the best understandings and greatest circumspection, are guilty of that fault in reasoning which the writers on logick call *the insufficient, or imperfect enumeration of parts, or cases:* insomuch that I will venture to assert, that this is the chief, and almost the only, source of the vast number of erroneous opinions, and those too very often in matters of great importance, which we are apt to form on all the subjects we reflect upon, whether they relate to the knowledge of nature, or the merits and motives of human actions. It must therefore be acknowledged, that that art which affords a cure to this weakness, or defect, of our understandings, and teaches us so to enumerate all the possible ways in which a given number of things may be mixed and combined together, that we may be certain that we have not omitted any one arrangement of them that can lead to the object of our inquiry, deserves to be considered as most eminently useful and worthy of our highest esteem and attention. And this is the business of *the art, or doctrine of combinations.*

Nor is this art or doctrine to be considered merely as a branch of the mathematical sciences. For it has a relation to almost every species of useful knowledge that the mind of man can be employed upon. It proceeds indeed upon mathematical principles in calculating the number of the combinations of the things proposed: but by the conclusions that are obtained by it, the sagacity of the natural philosopher, the exactness of the historian, the skill and judgment of the physician, and the prudence and foresight of the politician, may be assisted; because the business of all these important professions is but *to form reasonable conjectures* concerning the several objects which engage their attention, and all wise conjectures are the results of a just and careful examination of the several different effects that may possibly arise from the causes that are capable of producing them. And, I presume, it was from a sense of the great and general utility of this doctrine that several very eminent mathematicians have undertaken to treat of it in their public writings; and particularly Mr. Van Schooten (the learned commentator on Des Cartes's geometry), Mr. Leibnitz, Dr. Wallis, and Monsieur

Monfieur Preftet: fo that the reader is not to confider every thing he will meet with in this treatife as entirely new and of my invention. I have, however, made fome improvements on the fubject, and thofe too of confiderable importance, which I may juftly call my own: and particularly I have difcovered a general and eafy demonftration of the principal and moft remarkable property of the figurate numbers, to wit, " that of the proportion between the fum of any number of terms of a feries of figurate numbers of any order whatfoever to the fum of the fame number of terms all equal to the laft term of the feries;" upon which property many of the following propofitions in this book are founded: for of this property I believe no other writer has ever before given a demonftration.

Indeed, none of the tracts hitherto publifhed on this fubject, can be faid to contain a full and fatisfactory account of it. And therefore I have thought it would be agreeable to my readers to fee it here treated in a regular manner, from the firft and moft fimple principles on which it is founded, to the higher and more extenfive propofitions which have been built upon them, without being under the neceffity of referring to other books upon the fubject. But, though, for thefe reafons, I have laid down the very firft elements of the doctrine, and have endeavoured to demonftrate every thing as I went on, to the end that the chain of reafoning might be uniform and compleat, I have done it in as concife a manner as I could, and only as far as was neceffary to prepare the way to the fubfequent and more important parts of the book. The greater part of the treatife confifts of two principal heads, of which the firft contains the doctrine of *permutations*, and the fecond contains the doctrine of *combinations*; which is followed by a third branch, which fprings out of the two former, and treats of *permutations* and *combinations* joined together.

CHAP-

CHAPTER I.

CONCERNING PERMUTATIONS.

ARTICLE 1. BY the *permutations* of a number of things, I mean the ſeveral variations that may be made in their relative ſituations, or poſitions, or in the order in which they may be made to follow each other, while their number continues the ſame. So that, when it is propoſed to find in how many different ways a given number of things may be ranged, or diſpoſed, without omitting any of them, this is ſaid to be requiring the *number of their permutations.*

2. The things of which we are required to diſcover the number of permutations, may be either all diſtinguiſhed from each other by ſome plain mark, ſuch as a difference of ſhape or colour, as cubes from ſpheres, or black balls from white balls; or they may be exactly like each other, ſo as to be liable to be miſtaken one for another, as two ſpherical black balls of exactly the ſame ſize and weight. In the former caſe it will be proper to denote the ſeveral things by as many different letters of the alphabet; and in the latter caſe it will be convenient to denote ſo many of the things as are exactly like each other, by the ſame letter of the alphabet, repeated as often as any of the ſaid things which are like each other ſhall occur, as will be ſeen in the courſe of the following pages. We will firſt conſider the former of theſe caſes, or that in which all the things are diſtinguiſhed from each other.

The

The first Case of Permutations, in which all the things whose permutations are required to be assigned, are distinguished from each other.

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3. As it is obvious that the number of changes of position that may happen in a great number of things cannot be determined without first knowing the number of the like changes of position that may happen in all lesser numbers of them, it is manifestly necessary, in treating of this subject, to proceed in the synthetick method, and begin our reasonings from the first and most simple cases: which may be done as follows.

4. If there is only one thing to be arranged, which is denoted by the letter *a*, it can be taken, or ranged, only in one manner.

5. If there are two things clearly distinguished from each other, which are denoted by the letters *a* and *b*, it is evident that we may either place *a* before *b*, or *b* before *a*; so that there will be two different ways of arranging them, to wit, *a b* and *b a*; or, in other words, there will be two *permutations* of them. Q. E. I.

6. If there be three things distinguished from each other, and denoted by the three different letters *a*, *b* and *c*, it is evident that either of the three letters may be placed before the other two. Now, if *a* is placed first, the other two letters *b* and *c* may undergo two permutations, by what has been seen in the last article, and the three letters may be placed in these two positions, *a b c*, and *a c b*; and in like manner, if *b* is placed first, the other two letters *a* and *c* may undergo two permutations, and the three letters may be placed in the two following positions, to wit, *b a c* and *b c a*; and, lastly,

laftly, if *c* is placed firft, the other two letters *a* and *b* may undergo two permutations, and the three letters may be placed in the two following pofitions, to wit, *cab* and *cba*. Therefore the whole number of permutations which the order, or pofition, of the three letters, *a*, *b*, and *c* may undergo, is three times 2, or 6, to wit, *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*. Q. E. I.

7. In like manner, if there are four different things clearly diftinguifhed from each other, and denoted by the four different letters *a*, *b*, *c*, and *d*, it is evident that either of the four may be placed before the other three, and that, while each of them is placed firft, the other three may undergo 6 permutations, by what has been juft now fhewn in art. 6. Therefore the whole number of permutations which thefe four things, or letters, may undergo, will be four times 6, or 24. Q. E. I.

8. And, for the fame reafon, if there were five things denoted by the five different letters *a*, *b*, *c*, *d* and *e*, the number of their permutations would be five times as great as in the laft cafe; or would be 5 times 24, or 120. And in general, whatever be the number of things or letters, the number of permutations, or changes of pofition, which they may be made to undergo, will be equal to the product that arifes by multiplying the number of permutations of the next fmaller number of things by the given number of them. So that, if the whole number of things, or letters, be n, and the number of permutations in $n - 1$ things, or letters, be N, the number of permutations in all the n letters, will be equal $n \times N$. And hence arifes the following

Rule

Rule for discovering the whole number of permutations, or relative changes of position, which any given number n, *of things, may be made to undergo.*

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9. Let all the numbers 1, 2, 3, 4, 5, 6, 7, &c, in their natural order, beginning from unity, up to the given number n, of things, or letters, whose permutations are to be investigated, be multiplied one into the other; and the product $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times$ &c . . . $\times\, n$ will be the number of permutations that is required. Q. E. I.

10. It will be convenient sometimes to use a full point [.] instead of the common mark of multiplication $\times$; and then $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times$ &c. $\times\, n$ will be $=$ 1 . 2 . 3 . 4 . 5 . 6 . 7 . &c . n, or (because 1 has no effect in multiplication) $=$ 2 . 3 . 4 . 5 . 6 . 7 . &c . n; which will therefore be equal to the whole number of permutations, or changes of position, which n things may be made to undergo.

11. According to this rule, the number of permutations, or changes of position, which 7 different things may be made to undergo, is 2 . 3 . 4 . 5 . 6 . 7, or 5040. Thus, for example, the different changes that may be rung upon seven different bells is 5040. The multiplications of these numbers into each other will appear in the following table;

The

The Number of Things.	*The Number of Permutations, or Changes of Position.*
1	1
	2
2	2
	3
3	6
	4
4	24
	5
5	120
	6
6	720
	7
7	5040
	8
8	40,320
	9
9	362,880
	10
10	3,628,800
	11
	3 628 800
	36 288 00
11	39,916,800
	12
	79 833 600
	399 168 00
12	479,001,600

12. We

12. We may see by this table how very fast the number of permutations increases, as the number of things to be arranged becomes greater and greater. The four letters that compose the word *Roma* may be arranged in 24 different ways; but the six letters that compose the word *Romani* may be arranged in 720 different ways; and the seven letters that compose the word *Romanis* may be arranged in no less than 5040 different ways. We are now to consider the second case of permutations, in which some of the things to be arranged are exactly like others of them, so as not to be distinguished from them.

The Second Case of Permutations, in which some of the things, the permutations of which are required to be assigned, are like others of them, so as not to be distinguished from them.

---

13. If some of the things of which we are required to find the permutations, are exactly like others of them, so as not to be distinguished from them, the number of permutations, or changes of position, which they may be made to undergo, will be much smaller than in the former case. Thus, for example, if there are six different things, whereof we are required to assign the permutations, but three of them are exactly like each other, so that it is impossible to distinguish either of them from the other two; as is the case with the six letters *a a a b c d*, in which the letter *a* occurs three times; the number of permutations which these six things, or letters, can undergo, will be much less than the number of permutations they could undergo, if they were all distinguishable from each other, as they were supposed to be in the former case. And the way of finding out how much less the number of permutations will be in this case than in the former case, will be to consider how many per-

 mutations,

mutations, or changes of position, the three things which are exactly alike, and are denoted by the same letter *a*, might undergo, if they were unlike each other, and distinguishable one from the other, and then to substitute an unit, or one single position, in lieu of all those several permutations. Thus, for example, if, instead of the three things exactly alike which are denoted by the same letter *a*, we were to take three things that were unlike each other, and denote them by the three letters *a*, *α*, and a, that is, by an Italick *a*, a Greek *α*, and a Roman a, it is evident from what has been shewn in art. 6, that, without making any change in the position of the other letters, *b*, *c*, *d*, these three letters *a*, *α*, and a, might be placed in six different positions, instead of the one position *a a a* in which alone the three things perfectly alike, that were denoted by the same letter *a*, could be arranged. The number of permutations therefore in the six things denoted by the letters *a*, *α*, a, *b*, *c*, *d*, will be six times as great as that of the six things denoted by the letters *a*, *a*, *a*, *b*, *c*, *d*, in which three of the things are alike, and denoted by the same letter *a*. And therefore, to find the number of permutations of the six things denoted by the letters *a*, *a*, *a*, *b*, *c*, *d*, we must first find the whole number of permutations which they might undergo if they were all unlike each other, and denoted by the letters *a*, *α*, a, *b*, *c*, *d*, and then we must divide the said number by 6, or the number of permutations which the three things denoted by the same letter *a* might undergo if they were unlike each other, and denoted by the three different letters *a*, *α*, and a. Now the whole number of permutations of six different things unlike each other, that are denoted by the letters *a*, *α*, a, *b*, *c*, *d*, has been shewn to be 720. Therefore the number of permutations of six different things, whereof three are perfectly like each other,

and

and denoted by the ſame letter *a*, or of ſix different things denoted by the letters *a*, *a*, *a*, *b*, *c*, *d*, will be $\frac{720}{6}$, or 120 *.

14. Again, if the ſix letters whereof we were required to find the permutations, were *a a a b b c*, in which, beſides the letter *a*, which is repeated three times, the letter *b* is alſo

* The truth of this article may be made viſible to the eye in the following manner :

Let us (to avoid a great number of permutations, which would take up a great deal of room, and tend to confound the ſubject) ſuppoſe the three different letters a, *a*, and α, to be connected only with one more letter, to wit, *b*. Then, by art. 7, the whole number of permutations of theſe letters will be 24, to wit,

a, *a*, α, *b*,	a, *a*, *b*, α,	a, *b*, *a*, α,	*b*, a, *a*, α,
a, α, *a*, *b*,	a, α, *b*, *a*,	a, *b*, α, *a*,	*b*, a, α, *a*,
a, a, α, *b*,	*a*, a, *b*, α,	*a*, *b*, a, α,	*b*, *a*, a, α,
a, α, a, *b*,	*a*, α, *b*, a,	*a*, *b*, α, a,	*b*, *a*, α, a,
α, a, *a*, *b*,	α, a, *b*, *a*,	α, *b*, a, *a*,	*b*, α, a, *a*,
α, *a*, a, *b*,	α, *a*, *b*, a, .	α, *b*, *a*, a,	*b*, α, *a*, a.

Now, let the Italick letter *a*, and the Greek letter α, be converted into the Roman letter a. And the foregoing permutations will thereby be converted into the following ones, to wit,

a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a,
a a a *b*,	a a *b* a,	a *b* a a,	*b* a a a;

of which the firſt ſix are all exactly alike, to wit, a a a *b*, and therefore muſt be reckoned as only one poſition, or permutation ; and, in like manner, the next ſix are alſo all alike, to wit, a a *b* a, and therefore muſt be reckoned as only one poſition, or permutation ; and the third ſix are alſo all alike, to wit, a *b* a a, and therefore muſt be reckoned as only one poſition, or permutation; and laſtly, the fourth ſix are alſo all alike, to wit, *b* a a a, and therefore muſt be reckoned as only one poſition, or permutation. So that, by the coincidence of ſix permutations into one in each of the four ſets of ſix permutations, the ſaid twenty-four different permutations will be reduced to only four, or $\frac{24}{6}$, different permutations, to wit, a a a *b*, a a *b* a, a *b* a a, *b* a a a.

And it is eaſy to ſee that the like reduction muſt take in the whole number of permutations that may happen amongſt any other given number of things that are all different and diſtinguiſhable from each other, when any other and leſſer number of the ſaid things are rendered like to, and undiſtinguiſhable from, each other.

repeated

repeated twice, it is evident that the number of the permutations which the ſaid letters could undergo, would be but half the number of the permutations of the ſix letters *a a a b c d*; becauſe every two permutations of theſe letters which would be diſtinguiſhable from each other when the two different letters *b* and *d* are made uſe of, will coincide, or become undiſtinguiſhable from each other, when *b* is inſerted inſtead of *d*. And therefore the number of the permutations of the ſix letters *a a a b b c* will be only $\frac{120}{2}$, or 60.

15. And in the ſame manner it may be ſhewn that, when ſeveral of the letters, of which we are required to aſſign the number of permutations, are repeated, or taken more than once, we muſt, for every ſuch repetition of the ſame letter, divide the number of permutations of the whole number of letters by the number of the permutations of ſo many different letters as there are repetitions of the ſame letter. And hence ariſes the following

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Rule for diſcovering the number of permutations, or relative changes of poſition diſtinguiſhable from each other, which any given number n, *of things, whereof ſome are exactly like others, and cannot be diſtinguiſhed from them, may be made to undergo.*

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16. Let the whole number of permutations, or changes of poſition, which the ſaid things might be made to undergo, if they were all unlike each other, and could be clearly diſtinguiſhed one from the other, be divided by the number of permutations, or changes of poſition, which the two, or more, things which are like each other, and are denoted by the ſame letter, might be made to undergo, if they were

4 unlike

unlike to each other, and clearly diftinguifhed from each other. And the quotient will be the number of permutations that is required. This is upon a fuppofition that, amongft the things that are given, and of which we are required to find the number of permutations, there is only one fet of things that are exactly like each other, and therefore denoted by the fame letter.

But, if, amongft the things of which we are required to find the number of permutations, there fhould be two, or more, fets of things that are exactly like each other, and therefore denoted by the repetition of the fame letters, we muft multiply the number of all the permutations which the firft fet of like things, denoted by the firft letter that occurs more than once in the notation, might be made to undergo if they were all unlike each other, into the number of all the permutations which the fecond fet of like things, denoted by the fecond letter that occurs more than once in the notation, might be made to undergo if they were all unlike each other, and further into the number of all the permutations which the third fet of like things, denoted by the third letter that occurs more than once in the notation, might be made to undergo, if they were all unlike each other, and into the numbers of all the permutations which the fourth fet, and the fifth fet, and all the following fets, of like things, denoted by the repetition of the fame letters, might be made to undergo, if the things in each fet were unlike each other: and the whole number of permutations, which all the n things that are given (and whereof we are required to find the number of permutations diftinguifhable from each other) might be made to undergo, if they were all unlike each other, muft be divided by the product of the faid multiplication. The quotient will be the number of permutations diftinguifhable from each other, of the given number n of things, which was required to be found.

17. This doctrine of permutations is of great ufe in determining the number of anagrams that may be made of any propofed word, or the number of different ways in which the letters that compofe it may be arranged. Thus, for example,

ample, the letters that form the word *Roma* may be arranged in 2 . 3 . 4, or 24, different ways; and thoſe of the word *Romani* (which are ſix in number) may be arranged in 2 . 3 . 4 . 5 . 6, or 720, different ways; and thoſe of the word *Romanis* (which are ſeven in number) may be arranged in 2 . 3 . 4 . 5 . 6 . 7, or 5040, different ways; as we have ſeen in art. 12. In like manner the letters of the word *Trojanum* (which are eight in number) may be arranged in 2 . 3 . 4 . 5 . 6 . 7 . 8, or 40,320, different ways; and thoſe of the word *Doctrinam* (which are nine in number) may be arranged in 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9, or 362,880, different ways. But the letters of the word *Leopoldus*, though they are alſo nine in number, cannot be arranged in ſo many different ways, becauſe of the repetition of the letters *l* and *o*, each of which occurs twice. The number of different ways in which the letters of this word can be arranged is $= \frac{362,880}{2 . 2}$, or $\frac{362,880}{4}$, or 90,720; becauſe the two *l*s, if they were different letters, would admit of two permutations, and the two *o*s, if they were different letters, would likewiſe admit of two permutations, and conſequently theſe numbers of permutations, to wit, 2 and 2, muſt (according to the foregoing rule) be multiplied into each other, ſo as to make the product 4, and then the number 362,880 (which is the whole number of permutations which nine different letters may be made to undergo) muſt be divided by it, which gives the quotient 90,720. And the letters of the word *Studioſus*, though likewiſe nine in number, will admit of only 30,240 permutations, becauſe of the repetition of the letter *u* twice, and the letter *s* three times. For the permutations which the two *u*s might be made to undergo, if they were different letters, are 2, and the permutations which the three *s* es might be made to undergo, if they were different letters, is 6; and the product that ariſes by multiplying 2 into 6 is 12. We muſt therefore divide 362,880 (which is the whole number of permutations of nine different letters) by 12; and the quotient 30,240 will be the number of all the permutations of the nine letters of the word *Studioſus* that will be different, or diſtinguiſhable from each other.

18. It

18. It is only by the aſſiſtance of this doctrine of permutations that all thoſe queſtions can be determined, which ſome learned and ingenious men have propoſed concerning the number of the variations, or tranſpoſitions of the words contained in certain verſes, which, on account of the great number of ſuch tranſpoſitions which may be made in them, have been called *Proteus verſes*, in alluſion to the Egyptian ſea-god of that name mentioned in Homer's Odyſſey, who was ſo famous for aſſuming many different ſhapes. The moſt celebrated of theſe verſes are thoſe which have been given us by Thomas Lanſius, and the learned Joſeph Scaliger, and Bernard Bauhuſius, a Jeſuit of the college at Louvain, in the Auſtrian Netherlands. The following two verſes we have from Thomas Lanſius:

Lex, Rex, Grex, Res, Spes, Jus, Thus, Sal, Sol, (bona) *Lux, Laus:*
Mars, Mors, Sors, Lis, Vis, Styx, Pus, Nox, Fæx, (mala) *Crux, Fraus.*

In each of theſe verſes there are eleven words of one ſyllable, and one word of two ſyllables, to wit, *bona* in the firſt, and *mala* in the ſecond. Theſe two words of two ſyllables muſt always remain in the ſame place, or within two words of the end of the lines, in order to preſerve the meaſure of the verſes, which requires that the fifth foot in each verſe ſhould be a dactyl. But the other eleven words in each verſe may be placed in any order, with reſpect to each other, that we pleaſe, without altering the meaſure of the verſes. Now the number of permutations, or changes of poſition, that eleven different things can undergo is 39,916,800, as appears from the table in art. 11. It follows therefore that the words of each of the two foregoing verſes may be tranſpoſed in 39,916,800 different ways, without ſpoiling the meaſure of them.

19. In ſome other inſtances of theſe *Proteus* verſes that have been given by ingenious writers on this ſubject, it happens that many of the tranſpoſitions of the words contained

in them are incompatible with the meaſure of the verſes, and ſome of them, from the irregular and ungrammatical order in which the words follow each other, ſeem to convey no ſenſe or meaning whatſoever, or, perhaps, in ſome caſes, a different ſenſe from that which the author intended. But in all theſe caſes a little attention and care will enable us to diſtinguiſh the uſeful tranſpoſitions from the abſurd ones, and to determine the numbers of tranſpoſitions of each ſort ſeparately, if we proceed by regular ſteps according to ſome order, or plan of admiſſion or excluſion, in making the enquiry. An inſtance of this kind occurs in the following Hexameter Latin verſe, which was made by the abovementioned Bernard Bauhuſius, the Jeſuit of Louvain, in honour of the bleſſed Virgin Mary, the mother of our Saviour Jeſus Chriſt; to wit,

Tot tibi ſunt dotes, Virgo, quot ſidera cælo.

On this celebrated verſe ſeveral men of great learning and reputation have beſtowed a great deal of attention. For, in the firſt place, *Ericius Puteanus*, in a little book which he publiſhed under the title of *Thaumata Pietatis*, has employed no leſs than 48 pages in reckoning up the ſeveral uſeful, or rational, tranſpoſitions that may be made of the words contained in it, and makes them amount to as many at leaſt as there are ſtars in the heavens, the number of which is uſually ſaid to be 1022; leaving out (through a religious reverence for the character of the Virgin Mary) all thoſe tranſpoſitions which ſeem to affirm that there are as many ſtars in the heavens as there are virtues in the Virgin's character, becauſe he thinks the number of the latter to be much greater than that of the former. And, 2dly, Gerard Voſſius, in the 7th chapter of his treatiſe intitled, *De Scientiis Mathematicis*, has affirmed the number of the tranſpoſitions which may be made in the words of this verſe without ſpoiling the ſenſe or the meaſure, to be 1022, as *Puteanus* had made it before him. And, 3dly, Monſieur *Preſtet*, a French mathematician, in the firſt edition of a book called *The Elements of the Mathematicks*, page 348, has examined this *Proteus* verſe, and declared it to admit of 2196 tranſpoſitions

poſitions of its words without ſpoiling the ſenſe or the meaſure: and afterwards, in the ſecond edition of his ſaid work, vol. i. page 133, having re-conſidered the ſubject, has increaſed the number of theſe tranſpoſitions to almoſt half as many more, or 3276. And, 4thly, the induſtrious compilers of the Leipſic *Acta Eruditorum*, in the month of June, 1686, in giving an account of Dr. Wallis's Treatiſe of Algebra, have fixed the number of theſe tranſpoſitions (which Dr. Wallis himſelf had not in that treatiſe ventured to aſſign) at 2580. And, laſtly, Dr. Wallis himſelf, in a Latin edition of his works which he publiſhed afterwards in the year 1693, page 494, has carried the number of theſe tranſpoſitions to 3096. But all theſe writers have been miſtaken in their calculations, and have aſſigned wrong numbers for the ſolution of this queſtion; which cannot but ſeem rather ſurpriſing, as ſome of them had examined the ſubject twice over, and corrected their firſt concluſions. The true number of tranſpoſitions of its words which this famous hexameter verſe will admit of without ſpoiling either the ſenſe or the meaſure of it, that is, without admitting a ſpondee in the fifth place, but admitting ſuch tranſpoſitions as only deſtroy the *cæſura* of the verſe, I have found, upon a careful examination, to be 3312.

20. I here conclude the chapter on the doctrine of permutations, of which I hope the fundamental principles have been ſufficiently explained; and I proceed to conſider the doctrine of combinations, which is of no leſs uſe and importance than the former.

CHAPTER II.

CONCERNING COMBINATIONS.

DEFINITION I.

21. BY the *combinations* of things, I mean the ſeveral different ways in which any given number of things may be joined, or connected with each other, without any regard to their relative poſitions, or the order in which they follow one another. So that, when a certain number *n* of things is given, and we are required to find in how many different ways theſe *n* things may be taken, by taking, firſt, two of them at a time, then three of them at a time, then four of them at a time, and ſo on in all other poſſible conjunctions, ſo that no one heap, or parcel, of them ſhall be taken more than once, we are ſaid to be required to find all the poſſible *combinations* of the ſaid given number of things.

DEFINITION 2.

22. The number of the things given which is directed to be joined together in one heap, or parcel, is called *the exponent* of the combination : ſo that if we are directed to combine them by pairs, or in parcels containing two a-piece, the exponent of the combination will be 2 ; if we are directed to combine them by triplets, or in parcels containing three a-piece, the exponent of the combination will be 3 ; and if we are directed to combine them by quadruplets, or in parcels containing four a-piece, the exponent of the combination will be 4 ; and, in general, if we are directed to combine them in parcels containing *m* a-piece, the exponent of the combination will be the number *m*.

Definition 3.

23. And the feveral things combined in thefe different manners are called *pairs*, or *couplets*, and *triplets*, and *quadruplets*, &c, or *binaries*, *ternaries*, and *quaternaries*, &c, or *binions*, *ternions*, and *quaternions*, &c; that is, all the different conjunctions, or combinations, of any given number of things in parcels confifting of two things each, are called all the *pairs*, or *couplets*, or *binaries*, or *binions*, in the faid number of things; and all the different conjunctions or combinations of them in parcels confifting of three things each, are called all the *triplets*, or *ternaries*, or *ternions*, in them; and all the different conjunctions or combinations of them in parcels confifting of four things each, are called all the *quadruplets*, or *quaternaries*, or *quaternions*, in them. And fimilar names may be found for thefe combinations, when the number of things contained in a fingle parcel is greater than 4.

24. And when the things are taken fingly, or feparately, or one by one, it will be convenient to denominate them *unaries*, or *unities*, or to give them fome name that bears a refemblance to the names by which we diftinguifh the feveral combinations of them with each other in parcels of two, or of three, or of four, or more together: becaufe, though, when they are taken fingly, they cannot, in a ftrict fenfe, be faid to be *combined*, or the taking them fingly cannot, in ftrictnefs, be called a *combination* of them, yet in this doctrine of combinations it is often neceffary to take into confideration the number of them when taken fingly, in order to determine all the variations that can be made upon them; and therefore, in a loofer and more extenfive fenfe of the word *combinations*, the things, when taken feparately, are confidered as undergoing one fpecies of combination, or forming one clafs of the feveral claffes of combinations which they may be made to undergo. This is a fmall inaccuracy of language, fimilar to that by which a unit is often called a *number*, though in ftrictnefs a *number* means *two* or *more* *units*,

units, or single quantities, joined together. But when due notice is given of what is meant by such inaccurate expressions (which are often convenient for the purpose of avoiding a multiplicity of words) no mistakes can arise from the use of them.

25. And for a like reason it will be convenient to give a name to the act of omitting to take them at all, either singly, or combined with each other, or to consider such omission as one species of their combinations. Such an omission of them may be called a *nullienation* (from the word *nullies*, which means *no-times*); and the *nothings*, or cyphers, set down, instead of the things themselves, on these occasions may be called *nullenaries* (like *binaries*, *ternaries*, and *quaternaries*) or *nullions*. The use of this sort of odd language will appear in the course of the following pages.

26. Some writers have confined the word *combination* to the strict original sense of "taking things by *binaries*, or *pairs*, or *couplets*, only, or parcels consisting of two things;" and have called the taking them by *ternaries*, or parcels consisting of three things, *conternation*; and the taking them by *quaternaries*, or parcels consisting of four things, *conquaternation*; and have denominated the parcels consisting of two things each, that may be formed out of a given number of things, the *combinations* of the said given things; and the parcels consisting of three things each, which may be formed out of the same given number of things, the *conternations* of the said given things; and the parcels consisting of four things each, which may be formed out of the same given number of things, the *conquaternations* of the said given things. But this degree of accuracy in our expressions would evidently lead to the composition of an immense number of new words, in order to express the variety of conjunctions that may be made of the things given in parcels of different sorts, such as parcels consisting of two things, parcels consisting of three things, parcels consisting of four things, parcels consisting of five things, parcels consisting of six things, and the like; the use of which multitude of new words might

might be found inconvenient. And therefore other perfons, who were apprehenfive of this inconvenience, and yet were defirous of avoiding the inaccuracy of employing the word *combinations* for parcels confifting of more than two things, have propofed to make ufe of the more general words *complications* or *complexions* (derived from the Latin verb *complicare*, which fignifies *to fold up together)* for parcels confifting of three things, or four things, or five, or more, things, each, made out of a given number of things fucceffively fo united together: and fome authors, with great fagacity and judgment, have recommended the word *elections* to be ufed on this occafion, in order to comprehend thofe methods of reckoning and claffing the things under confideration by which the things when taken feparately, or one by one, are admitted as one fpecies of combinations of them; and even, when nothings, or cyphers, are taken in their ftead, thofe nothings, or cyphers, are admitted as another fpecies of their combinations, or elections. But the generality of writers who have treated of this fubject, make ufe of the word *combinations* to denote all the different parcels of things, whether confifting of two things, or of three things, or of four things, or of any greater number of things, which can be formed out of a given parcel of things; and even to denote the given things, when taken fingly or feparately; and alfo the nothings, or cyphers, which are fet down inftead of them, when they are not taken at all: nor does there feem to be any neceffity for inventing new words on the occafion.

Thefe definitions of the words that will occur moft frequently in this doctrine of combinations being premifed, I now proceed to confider the doctrine itfelf.

27. Now when we are enquiring into the number of combinations of a given number of things, the faid things may either be all unlike and clearly diftinguifhable from each other, or fome of them may be exactly like others of them, fo as not to be diftinguifhable from them. And the faid things may either be fo combined together that no one thing fhall be contained oftener in any of the propofed combinations than it occurs in the original number of things which are propofed

propoſed to be combined together; or they may be combined together without this reſtriction, or ſo that in ſome of the propoſed combinations the ſame thing may occur oftener than it does in the original number of things which are propoſed to be combined together, to wit, by being combined with itſelf *. And different ſuppoſitions may be made, and different

* I am not quite certain that this laſt ſentence is a faithful tranſlation of the original, which I ſhall therefore here ſubjoin for the reader's attentive conſideration. *Eæque vel ita combinari debent, ut in nullâ combinatione res eadem ſæpiùs contineatur quàm ipſa reperitur in toto rerum numero; vel ſic, ut in eâdem combinatione res eadem etiàm ſæpiùs recurrere, hoc eſt, ut ſecum ipſâ quoque combinari, poſſit.* The meaning of this obſcure ſentence (as far as I can underſtand it) may be illuſtrated by the two following examples.

In the firſt place, let us ſuppoſe that the things that are to be combined together are ſix in number, all clearly diſtinguiſhable from each other, and denoted by the ſix letters *a*, *b*, *c*, *d*, *e*, *f*. And let us ſuppoſe that theſe ſix letters are to be combined together in quaternions, or quadruplets, or parcels conſiſting of four letters each. Then, ſays the author, theſe quadruplets may be either reſtrained to thoſe only which conſiſt of four different letters, or in which the ſame letter does not occur oftener than once, or than it occurs in the original enumeration of the ſix things, *a*, *b*, *c*, *d*, *e*, *f*, out of which theſe quadruplets are to be formed, ſuch as the quadruplets *a b c d*, *a c d e*, *a d e f*, &c; or theſe quadruplets may be formed without this reſtriction, ſo as to admit the ſame letter to be contained in them, more than once, or to be, as it were, combined with itſelf, as happens in the quadruplets *a a b c*, *a a c d*, *a a d e*, *a a e f*, *a a b d*, *a a b e*, *a a b f*, *a a c e*, *a a a b*, *a a a c*, *a a a d*, &c.

In the ſecond place, let us ſuppoſe that the things that are to be combined together are, as before, ſix in number, but that two of them are exactly like each other, and are therefore denoted by the ſame letter *a*, and that three of them are alſo exactly like each other, and therefore denoted by the ſame letter *b*, and the ſixth is different both from thoſe of the firſt ſet, and from thoſe of the ſecond ſet, and is therefore denoted by the letter *c*; ſo that the ſix things that are to be combined together, are denoted by the letters *a*, *a*, *b*, *b*, *b*, *c*. And let it be required to combine theſe ſix things, or letters, together in quaternions, or quadruplets, or parcels conſiſting of four letters each. Then, ſays the author, theſe quadruplets may be either ſo reſtrained that they ſhall not contain either of the three letters *a*, *b*, and *c* oftener than it is contained in the original enumeration of them, to wit, *a*, *a*, *b*, *b*, *b*, *c*, that is, that no quadruplet ſhall contain the letter *a* oftener than twice, or the letter *b* oftener than three times, or the letter *c* oftener than once; as is the caſe with the quadruplets *a b b c*, *a a b c*, *a a b b*, *a b b b*, *b b b c*; or the ſaid quadruplets may be formed without this reſtriction, ſo as to admit the letter *a* to be repeated more than twice, and the letter *b* to be repeated more than three times, and the letter *c* to be repeated more than different queſtions conſequently may be propoſed, concerning the manner in which the quantities are combined together. For it may either be required to find the number of all the poſſible combinations of a given number of things, by taking them firſt ſingly, then in couplets or parcels of two, and then in triplets or parcels of three, and then in quadruplets or parcels of four, and ſo on according to all the exponents they will admit of; or it may be required only to find the number of all the combinations that may be made of the ſame given number of things according to one, or more, of their exponents, ſeparately; as, for inſtance, by diſpoſing them in parcels of two things, or in parcels of three things, or in parcels of four things, each. In each of theſe ways of combining the things under conſideration, a great variety of queſtions and problems may be propoſed concerning them, the full diſcuſſion of which would lead us into a very ample field of ſpeculation. But of theſe we ſhall only ſelect a few of the moſt curious and important, which we conceive to be neceſſary to the ſolution of the queſtions concerning the doctrine of chances, or the art of forming reaſonable conjectures concerning future events depending on chance, which will be conſidered in the ſubſequent part of this treatiſe.

28. Let it then be required, in the firſt place, to find the number of all the poſſible combinations that can be formed of a given number of things according to all the exponents they will admit of, upon a ſuppoſition that all the things that are to be combined together, are unlike to, and clearly diſtinguiſhable from, each other, and conſequently are denoted by different letters.

Let the things that are to be thus combined be denoted by the ſeveral ſmall letters *a*, *b*, *c*, *d*, *e*, &c. Let theſe

than once; as is the caſe in the quadruplets *aaab*, *aaac*, *aaaa*, *bbbb*, *aacc*, *accc*, &c.

This is the only meaning that I can find for the foregoing paſſage; but I cannot help entertaining ſome doubt whether it is the true one. *Ideò quære.*

letters be fet down in feparate lines, or rows, one under another, in the manner following.

In the firft line we muft place the firft letter *a*, by itfelf.

In the fecond line we muft place the letter *b*; firft, by itfelf; and then in conjunction with *a*, fo as to form the combination, or couplet, *a b*, or *b a*. For *a b* and *b a* are, in this doctrine of combinations, to be confidered as only one combination, becaufe in this doctrine no regard is to be had to the order in which the letters are placed, as there was in the doctrine of permutations.

In the third line we muft place the third letter *c*; firft, by itfelf; and then in conjunction with the preceeding letters *a* and *b*, fo as to form the binions, or couplets, *a c*, *b c*; and, laftly, with the preceeding couplet *a b*, fo as to form the triplet *a b c*.

a.

b, ab.

c, ac, bc, abc.

d, ad, bd, cd, abd, acd, bcd, abcd.

e, ae, be, ce, de, abe, ace, bce, ade, bde, cde, abce, abde, acde, bcde, abcde.

In the fourth line we muft place the fourth letter *d*; firft, by itfelf; and, 2dly, in conjunction with each of the three preceeding letters *a*, *b*, and *c*, fo as to form the three pairs or couplets of letters, *a d*, *b d*, and *c d*; and, 3dly, in conjunction with each of the three foregoing couplets, *a b*, *a c*, *b c*, fo as to form the three triplets *a b d*, *a c d*, and *b c d*; and, 4thly, in conjunction with the foregoing triplet, *a b c*, fo as to form the quadruplet, *a b c d*.

And in like manner the fifth letter *e* muft be placed in the beginning of the fifth line; firft, by itfelf; and, 2dly, in conjunction with each of the four preceeding letters *a*, *b*, *c*, *d*, fo as to form the four pairs, or couplets, *a e*, *b e*, *c e*, and *d e*; and, 3dly, in conjunction with each of the fix foregoing couplets,

couplets, *a b*, *a c*, *b c*, *a d*, *b d*, *c d*, ſo as to form the ſix triplets *a b e*, *a c e*, *b c e*, *a d e*, *b d e*, and *c d e*; and, 4thly, in conjunction with each of the foregoing triplets, *a b c*, *a b d*, *a c d*, *b c d*, ſo as to form the four quadruplets *a b c e*, *a b d e*, *a c d e*, and *b c d e*; and, 5thly, in conjunction with the foregoing quadruplet *a b c d*, ſo as to form the quintuplet, *a b c d e*.

And in the ſame manner muſt every following letter, *f*, *g*, *h*, &c, be combined with each of the preceeding letters, and with every preceeding combination of them, if the number of things, or letters, to be combined together, was more than five.

29. And from this manner of combining any given number of things, or letters, together, it is plain that we ſhall thereby obtain all the poſſible combinations of them, ſo that no combination of them whatſoever can be formed, or conceived, that will not be contained in one or other of the ſucceſſive lines, or rows, of quantities ſo generated from each other: and likewiſe it is plain that each of the combinations ſo obtained will be different from every other, or that no combination will occur in the ſaid lines more than once. And conſequently the ſum-total of all the combinations ſet down in the lines, or rows, of quantities ſo formed out of any given number of quantities, will be the number of all the poſſible combinations which the ſaid given number of quantities will admit of. We muſt therefore endeavour to find the number of all the combinations of a given number of quantities that will be contained in an equal number of lines, or rows, of quantities formed, or generated, from each other in the manner above deſcribed.

30. Now, in order to diſcover the number of the combinations contained in a given number of the foregoing lines, or rows, of quantities, it will be proper to obſerve, " that every new line, or row, muſt contain as many combinations as all the preceeding lines, or rows, added together, and one combination over;" and for this reaſon, to wit, becauſe the letter which is at the beginning of every new line, is placed

 there,

there, first, by itself, and afterwards in conjunction with all the letters and their several combinations in all the preceeding lines. Thus, for example, the letter e is placed in the beginning of the fifth line, first by itself, and afterwards in conjunction with each of the foregoing letters a, b, c, d, and with every combination of them in couplets, triplets, and quadruplets, contained in all the four foregoing lines; and consequently the number of combinations contained in this fifth line (reckoning the letter e by itself for one of them) will be equal to the number of all the combinations contained in all the four preceeding lines, beginning with the letters a, b, c, and d, and one combination over. This observation is of great use. For from it we may deduce the whole number of combinations contained in any given number of these lines, or which may be made with any given number of letters, by reasoning in the manner following.

A Lemma.

31. If in the increasing geometrical progression $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 +$ &c, of which the first term is 1, and the common ratio is that of 1 to 2, we take any number of terms whatever, as, for example, n terms, and call the sum of the said n terms S, and afterwards add another term to the said series, the said new term will be equal to $S + 1$, or to the sum of all the former n terms together with an unit.

Demonstration.

Since the terms in the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 +$ &c, increase continually in the proportion of 1 to 2, it is evident that all the terms of it after the first term 1 will be the several powers of 2 in their natural order, to wit, $\overline{2|}^{1}$, $\overline{2|}^{2}$, $\overline{2|}^{3}$, $\overline{2|}^{4}$, $\overline{2|}^{5}$, $\overline{2|}^{6}$, $\overline{2|}^{7}$, &c, and conſequently that the nth, or laſt, term of it will be $\overline{2|}^{n-1}$. If therefore to theſe n terms we add another term, the ſaid new term will be $= 2 \times \overline{2|}^{n-1}$, or $\overline{2|}^{n}$. We are therefore to ſhew that $\overline{2|}^{n}$ is $= S + 1$.

Now, ſince S is $= 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c + \overline{2|}^{n-1}$, 2 S will be $= 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + \&c + \overline{2|}^{n}$; which ſeries conſiſts of n terms as well as the ſeries $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c + \overline{2|}^{n-1}$, or S. Therefore (adding 1 to both ſides) $2S + 1$ will be equal to the ſeries $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + \&c + \overline{2|}^{n}$, that is, to the ſeries $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + \&c + \overline{2|}^{n-1}$ together with the new term $\overline{2|}^{n}$, or to $S + \overline{2|}^{n}$. Therefore (ſubtracting the ſeries S from both ſides) the new term $\overline{2|}^{n}$ will be equal to $S + 1$, or to the ſum of all the n terms of the ſeries $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c + \overline{2|}^{n-1}$, together with 1. Q. E. D.

Coroll. 1. It follows therefore that the ſecond and third and other following terms of the increaſing geometrical progreſſion $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + \&c$, may be generated, or derived, from the firſt term 1, not only by doubling it continually, but by the application of the property that has juſt now been ſhewn to belong to the terms of ſuch a ſeries, to wit, by adding together all the preceeding terms, and increaſing their ſum by an unit. Thus, $1 + 1$ will be $= 2$, which is the ſecond term; and $1 + 2 + 1$, will be $(= 3 + 1) = 4$, which is the third term; and $1 + 2 + 4 + 1$, will be $(= 3 + 4 + 1 = 7 + 1) = 8$, which is the fourth term; and $1 + 2 + 4 + 8 + 1$ will be $(= 7 + 8 + 1 = 15 + 1) = 16$, which is the fifth term; and $1 + 2 + 4 + 8 + 16 + 1$ will be $(= 15 + 16 + 1 = 31 + 1) = 32$, which is the ſixth term; and 1 +

2 + 4

2 + 4 + 8 + 16 + 32 + 1 will be (= 31 + 32 + 1 = 63 + 1) = 64, which is the ſeventh term; and 1 + 2 + 4 + 8 + 16 + 32 + 64 + 1 will be (= 63 + 64 + 1 = 127 + 1) = 128, which is the eighth term. And in the ſame manner may all the following terms of this progreſſion, however numerous, be generated from thoſe that preceed them by means of the aforeſaid property.

Coroll. 2. And hence it follows, *è converſo*, that, if there be a ſeries of terms beginning from 1, the terms of which are generated one from the other by means of the foregoing property, or by adding together all the preceeding terms, and increaſing their ſum by an unit, the ſaid ſeries will be the geometrical progreſſion 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c.

32. Now it has been ſhewn above in art. 28, that the numbers of quantities, or combinations, contained in the above-mentioned lines, or rows, beginning with the letters *a*, *b*, *c*, *d*, *e*, &c, are generated from the firſt quantity *a*, and from each other in the manner juſt now deſcribed, or that the number of quantities in every new line is equal to the ſum of the numbers of all the quantities in all the preceeding lines, together with an unit. It follows therefore, from coroll. 2, of the foregoing lemma, that the numbers of quantities contained in the ſaid ſeveral lines muſt conſtitute the geometrical progreſſion 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c, or 1 + $\overline{2}|^{1}$, + $\overline{2}|^{2}$, + $\overline{2}|^{3}$, + $\overline{2}|^{4}$, + $\overline{2}|^{5}$, + $\overline{2}|^{6}$, + $\overline{2}|^{7}$, + &c + $\overline{2}|^{n-1}$, ſuppoſing the number of lines to be *n*. Therefore the ſum of the numbers of quantities contained in all the *n* lines beginning with the letters *a*, *b*, *c*, *d*, *e*, &c, will be equal to the ſum of the firſt *n* terms of the increaſing geometrical progreſſion 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c, or to the ſeries 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + &c + $\overline{2}|^{n-1}$, or 1 + $\overline{2}|^{1}$, + $\overline{2}|^{2}$, + $\overline{2}|^{3}$, + $\overline{2}|^{4}$, + $\overline{2}|^{5}$, + $\overline{2}|^{6}$, + $\overline{2}|^{7}$, + &c + $\overline{2}|^{n-1}$. But, by the foregoing lemma, this ſeries together with an unit is equal

equal to $\overline{2}|^{n}$. Therefore this series alone is equal to $\overline{2}|^{n} - 1$. And consequently the number of all the quantities contained in all the said n lines, or the number of all the possible combinations of the n letters a, b, c, d, e, &c, (reckoning the said letters, when taken singly, among the said combinations) will be $= \overline{2}|^{n} - 1$. And hence arises the following

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Rule for finding the number of all the possible different combinations of a given number of things according to all their different exponents.

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33. Raise the number 2 to the power of which n or the given number of things, that are to be combined together, is the index; and subtract 1 from the said power. The remainder $\overline{2}|^{n} - 1$ will be the number of combinations that was required.

34. Coroll. 1. From this rule it follows, in the first place, that, if we consider the total omission of all the n letters as one way of combining them, the number of all the possible combinations will be greater than it was before by an unit, and therefore will be $= \overline{2}|^{n}$; and it follows, in the second place, that, if we exclude this case of the omission of all the letters (which may be called the *nullion*, or the combination of them by *nones*), and likewise exclude the several letters when taken singly, or separately, (which are not in strictness *combinations* of them), the number of the remaining combinations of n different things, or letters, in binions and ternions and quaternions, or in couplets, and triplets, and quadruplets, and in parcels of more than four letters in each, will be $\overline{2}|^{n} - 1 - n$, or $\overline{2}|^{n} - n - 1$.

Thus,

Thus, for example, the number of all the different conjunctions, or combinations, that may be made of the seven planets, to wit, Saturn, Jupiter, Mars, Venus, Mercury, the Earth, and the Moon, (taking the word *combinations* in the extent given to it in the foregoing rule) will be $= \overline{2}|^{7} - 1$ ($= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 - 1 = 128 - 1$) $= 127$; from which if we subtract 7, which is the number of the planets taken singly, or separately, (in which cases there are not properly any conjunctions of them), the remaining number $\overline{2}|^{7} - 1 - 7$, or $127 - 7$, or 120, will be the number of all the possible conjunctions, or combinations, of them by conjoining two together, or three together, or four together, or five together, or six together, or all the seven together; or, it will be the number of all the possible conjunctions of them properly so called. And the twelve registers, as they are called, or rows of pipes in a musical organ, by means of which the sound of it is made to change so remarkably from a soft and gentle sound to a very loud and solemn one, may be made to undergo $\overline{2}|^{12} - 1$, or $4096 - 1$, or 4095 combinations, or variations.

35. Corroll. 2. If we examine the number of combinations of the letters *a*, *b*, *c*, *d*, and *e*, in the several lines, or rows of quantities, set down above in art. 28, we shall find that the number of combinations that have even numbers for their exponents contained in each of the said lines after the first line (which contains only the single quantity *a*) is equal to the number of combinations that have odd numbers for their exponents contained in the same line. Thus, in the second line, which begins with the letter *b*, there is one quantity, namely *b*, with an odd number, to wit, 1, for its exponent, and one quantity, namely, *ab*, with an even number, to wit, 2, for its exponent. And in the third line beginning with the letter *c*, there are two quantities, namely, *c* and *abc*, with odd numbers, to wit, 1 and 3, for their exponents, and two quantities, namely, *ac* and *bc*, with an even number, to wit, 2, for their exponent. And in the fourth line beginning with the letter *d*, there are four quantities, namely, *d*, *abd*, *acd*, and *bcd*, which have the odd

numbers,

numbers 1 and 3 for their exponents; and there are four other quantities, namely, *a d*, *b d*, *c d*, and *a b c d*, which have the even numbers 2 and 4 for their exponents. And in the fifth line, beginning with the letter *e*, there are eight quantities, namely, *e*, *a b e*, *a c e*, *b c e*, *a d e*, *b d e*, *c d e*, and *a b c d e*, which have the odd numbers 1, 3, and 5 for their exponents; and eight other quantities, namely, *a e*, *b e*, *c e*, *d e*, and *a b c e*, *a b d e*, *a c d e*, *b c d e*, which have the even numbers 2 and 4 for their exponents. And the fame thing muft happen in the fixth line of quantities beginning with the letter *f*, and in every following line, becaufe every new line is formed by fetting down the new letter firft by itfelf, and then combining it with the firft letter *a*, and afterwards with all the quantities contained in the fecond, third, and other following lines that preceed the new line. Now the combination of the new letter with each of the quantities in the fecond, third, and other following lines, will turn all the quantities that have odd numbers for their exponents into quantities that have even numbers for their exponents, and all the quantities that have even numbers for their exponents into quantities that have odd numbers for their exponents. And therefore, as the number of quantities with odd numbers for their exponents in each of the faid fecond, third, and other following lines, was equal to the number of quantities with even numbers for their exponents, it follows that of the new quantities in the new line arifing from the combination of the new letter with all the quantities contained in the fecond, third, and other following lines, there will be as many that have odd numbers for their exponents as there will be that have even numbers for their exponents. And, if we add to thefe quantities the new letter itfelf, which is to be placed in the beginning of the new line, and of which the exponent is 1, and the combination of the new letter with the firft letter *a*, of which combination the exponent is the even number 2, whereby we fhall obtain all the quantities fet down in the new line, it is evident that the addition of thefe two quantities (of which the firft has the odd number 1, and the fecond has the even number 2, for its exponent) will not alter the equality of the numbers of com-

 binations,

binations, or quantities, of each kind, but that the number of quantities in the new line that have odd numbers for their exponents will ſtill be equal to the number of quantities in the ſame line that have even numbers for their exponents.

36. Coroll. 3. If therefore we add all the quantities contained in all the lines except the firſt line (which contains only the ſingle quantity *a*) together, it is evident that the number of quantities in ſuch ſum that will have odd numbers for their exponents will be equal to the number of quantities that will have even numbers for their exponents.

37. Coroll. 4. And, if we add together all the quantities contained in all the lines, including the firſt line, which contains the ſingle quantity *a* (the exponent of which is the odd number 1), the number of quantities in ſuch ſum that will have odd numbers for their exponents will exceed by an unit the number of quantities that will have even numbers for their exponents.

38. Coroll. 5. And, if to all the quantities contained in all the lines together we add, as another combination, the caſe denoted by a cypher 0, or the caſe of the omiſſion of all the letters, which we have above in art. 34, called the combination by *nones*, or the *nullion*, and conſider 0, or the exponent of this combination as an even number, the number of quantities in the ſaid ſum that will have odd numbers for their exponents will be exactly equal to the number of quantities that will have even numbers for their exponents.

39. Coroll. 6. It has been ſhewn in Coroll. 1, that the number of all the quantities in all the *n* lines taken together, and the caſe of the nullion is equal to $\overline{2}|^{n}$. It follows therefore from Coroll. 5, that the number of quantities in this ſum that will have odd numbers for their exponents will be equal to half of $\overline{2}|^{n}$, or to $\frac{\overline{2}|^{n}}{2}$, or $\overline{2}|^{n-1}$, and the number of

quantities

quantities in the same sum that will have even numbers for their exponents, including the nullion, will likewise be $\overline{2}|^{n-1}$; and consequently the number of quantities in the said sum that will have even numbers for their exponents, without including the nullion, will be $\overline{2}|^{n-1} - 1$. The same thing will be demonstrated in another manner here below in the 6th Corollary of Chapter 4.

CHAPTER III.

Of the numbers of combinations that may be made of a given number of things in parcels consisting of two things, or of three things, or of four thiugs, or of any other particular number of things, each; and of the numbers known by the name of the figurate numbers, *and their properties, with which the investigation of the said combinations is connected.*

40. FROM an attentive consideration of the five lines, or rows, of quantities in art. 28, of the foregoing chapter, beginning with the letters *a*, *b*, *c*, *d*, and *e*, and which exhibit all the different combinations that can be formed out of those five letters, it will be evident, that the *binions*, or *couplets*, or parcels consisting of two letters, in every new line of quantities, are formed by combining the single letter which is placed in the beginning of such new line, with each of the single letters contained in all the foregoing lines of quantities; and that the *ternions* or *triplets*, or parcels consisting of three letters, in such new line, are formed by combining the single letter which is placed in the beginning of such new line, with each of the *binions* or *couplets*, contained in all the foregoing lines; and the *quaternions*, or *quadruplets*, or parcels consisting of four letters,

 in

in such new line, are formed by combining the single letter which is placed in the beginning of such new line, with each of the *ternions*, or *triplets*, contained in all the foregoing lines; and, in like manner, in all higher combinations than *quaternions*, the combinations denoted by any exponent m in such new line, are formed by combining the said single letter which is placed in the beginning of such new line, with all the combinations denoted by the next lower exponent $m - 1$ contained in all the foregoing lines. It follows therefore that the number of *binions*, or *couplets*, of letters in every new line will be equal to the number of all the single letters in all the foregoing lines taken together; and that the number of ternions, or triplets, of letters in such new line, will be equal to the number of all the binions, or couplets, of letters in all the foregoing lines taken together; and the number of quaternions, or quadruplets, of letters in such new line, will be equal to the number of all the ternions, or triplets, of letters in all the foregoing lines taken together; and in like manner, that the number of combinations of any higher order than quaternions, denoted by the exponent m, in such new line, will be equal to the number of combinations of the next lower order, which is denoted by the exponent $m - 1$, in all the foregoing lines taken together. From these observations we may derive the following conclusions:

41. *First Conclusion.* As there is only one single letter in each of the said lines, or rows, of quantities, to wit, the letter in the beginning of the line, the single letters in all the lines successively will exhibit a set of units, to wit, 1, 1, 1, 1, 1, 1, 1, 1, &c, which are the figurate numbers of the first order.

42. *Second Conclusion.* As there is no binion, or couplet, of letters in the first line (which contains only the letter a), and there is only one binion in the second line, to wit, ab; and two binions in the third line, to wit, ac, bc; and three binions in the fourth line, to wit, ad, bd, cd; and, in general, as the number of binions in every new line is

 equal

equal to the number of the single letters in all the preceeding lines taken together; it follows that the numbers of binions, or couplets, of letters in the first, second, third, and fourth, and other following lines, will be 0, 1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1, &c, or 0, 1, 2, 3, 4, 5, &c, or the series of numbers 1, 2, 3, 4, 5, &c, in their natural order, with a cypher, 0, prefixed to them.

These numbers form an arithmetical progression, in which the common difference of the terms is 1; and they are often called *the natural numbers*, or a series of *lateral* numbers, or the figurate numbers of the second order.

43. *Third Conclusion.* As there are no *ternions*, or *triplets*, of letters in the two first lines; and there is one ternion, or triplet, to wit, *a b c*, in the third line; and 1 + 2, or 3 ternions, to wit, *a b d*, *a c d*, and *b c d*, in the fourth line; and 1 + 2 + 3, or 6 ternions, to wit, *a b e*, *a c e*, *b c e*, *a d e*, *b d e*, *c d e*, in the fifth line; and, in general, there are as many ternions in every new line as there are binions in all the foregoing lines together; it follows that the numbers of ternions, or triplets, of letters in the first, second, third, fourth, fifth, and other following lines, of rows, of quantities, will be 0, 0, 1, 3, 6, 10, 15, 21, 28, &c, which are formed by the continual addition of the numbers of the binions contained in the said lines, or of the terms of the series 0, 1, 2, 3, 4, 5, 6, 7, &c, with a new cypher, 0, prefixed to them.

These numbers 0, 0, 1, 3, 6, 10, 15, 21, 28, &c, are often called the *trigonal*, or *triangular*, numbers, or the figurate numbers of the third order.

44. *Fourth Conclusion.* As there are no quaternions, or quadruplets, in the three first lines; and there is one quaternion, or quadruplet, to wit, *a b c d*, in the fourth line; and 1 + 3, or 4, quaternions, to wit, *a b c e*, *a b d e*, *a c d e*, and *b c d e*, in the fifth line; and as, in general, there are as many quaternions in every new line as there are ternions in all the foregoing lines together; it follows that the number of quaternions, or quadruplets, of letters in the first, second, third,

third, fourth, fifth, and other following lines, or rows, of quantities, will be 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, &c, refpectively; which numbers are formed by the continual addition of the ternions, or triplets, contained in the faid lines, or of the terms of the laft preceeding feries 0, 0, 1, 3, 6, 10, 15, 21, 28, &c, with a new cypher, 0, prefixed to them.

Thefe numbers 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, &c, are often called the *pyramidal numbers*, or the figurate numbers of the fourth order.

45. *Fifth Conclufion.* In like manner the numbers of the quinions, or quintuplets, of letters contained in the feveral fucceffive lines, or rows, of quantities beginning with the letters *a*, *b*, *c*, *d*, *e*, &c, will be 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, &c, refpectively; which are formed by the continual addition of the quaternions, or quadruplets, contained in the faid lines, or of the terms of the laft preceeding feries 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, &c, with a new cypher, 0, prefixed to them.

Thefe numbers 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, &c, are often called the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, or the figurate numbers of the fifth order. And they are alfo fometimes called the *triangulo-triangular*, or *trigono-trigonal*, numbers.

46. *Sixth Conclufion.* And, in like manner, the numbers of the *fenions*, or *fextuplets*, of letters contained in the faid feveral fucceffive lines, or rows, of quantities beginning with the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, &c, will be 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, &c, refpectively; which are formed by the continual addition of the quinions, or quintuplets, contained in the faid lines, or of the terms of the laft preceeding feries 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, &c, with a new cypher, 0, prefixed to them.

Thefe numbers 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, &c, are often called the *pyramido-pyramidal* numbers, and fometimes the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, or the figurate numbers of the fixth order.

47. And

47. And in the ſame manner the numbers of the *ſeptenaries*, or *ſeptuplets*, and *octonaries*, or *octuplets*, and other higher combinations of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, &c, contained in the ſaid ſeveral ſucceſſive lines, or rows, of quantities beginning with the ſaid letters reſpectively, will form the ſeventh and eighth and other following higher orders of the figurate numbers reſpectively.

48. And thus we have unexpectedly been led by the conſideration of the nature of combinations to the contemplation of the *figurate* numbers, or of the numbers that are formed from a ſeries of equal numbers, or units, to wit, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c, by the continual addition of all the terms, and by the like addition of all the terms of every following ſeries ſo obtained. For theſe are the numbers to which arithmeticians have given the name of the *figurate numbers*.

49. In order to repreſent the ſeveral orders, or ſerieſes, of theſe figurate numbers in one view, and thereby to render what I have further to obſerve concerning them more eaſy to be apprehended, I have ſubjoined the following table of them, containing the firſt twelve terms of the firſt twelve orders, or ſerieſes, of the ſaid numbers; which the reader, if he chooſes it, may eaſily continue to a greater extent, both downwards, or towards the bottom of the page, and ſideways towards the right hand. In this table the Indian, or Arabian, figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 (that are placed on the left ſide of the table in a direction parallel to the ſide of the page, and ſeparated from the table by a double black line) expreſs the places, or numbers, of the ſeveral horizontal rows of numbers to which they are adjacent reſpectively, and alſo the numbers of letters, or things that are to be combined together. And the capital Roman figures I, II, III, IV, V, VI, VII, VIII, IX, X, XI, and XII, (that are placed directly over the table, parallel to the top of the page) expreſs the places, or numbers, of the ſeveral vertical columns, and are likewiſe the exponents of the combinations of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, &c, which

which are repreſented by the ſaid vertical columns reſpectively. And the ſaid vertical columns themſelves are the ſeveral orders, or ſerieſes, of figurate numbers, or of the ſeveral combinations of the firſt, ſecond, third, and other following orders, as far as the twelfth order, of which the Roman numerals I, II, III, IV, V, VI, VII, VIII, IX, X, XI, and XII, at the top of the table, are the exponents, with the proper number of cyphers, 0, prefixed to them. Thus, the firſt vertical column on the left-hand ſide of the table, under the Roman numeral I, is a ſeries of units, to wit, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, or the firſt order of figurate numbers, which Dr. Wallis calls *Monadicks*, and repreſents the numbers of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, that occur ſingly, or without being joined with any other letter, in the ſeveral lines, or rows, of quantities that are ſet down in chap. 2, art. 28, and which are ſuppoſed to be continued to the twelfth line; and the ſecond vertical column on the left-hand ſide of the table, under the Roman numeral II, is the ſeries 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or the ſecond order of figurate numbers called *the natural*, or *lateral*, numbers *, with a cypher, 0, prefixed to them, and repreſents the numbers of binions, or couplets, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by two in a parcel, that occur in the ſaid twelve ſucceſſive lines, or rows, of quantities; and the third vertical column, under the Roman numeral III, is the ſeries 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, or the third order of figurate numbers, called the *trigonal*, or *triangular*, numbers, with two cyphers prefixed to them, and repreſents the numbers of ternions, or triplets, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by three in a parcel, that occur in the ſaid twelve ſucceſſive lines: and the fourth vertical column, under the Roman numeral IV, is the ſeries 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, 120, 165, or the fourth order of figurate numbers (called the *pyramidal numbers*) with three cyphers prefixed to them, and repre-

* See Dr. John Wallis's *Diſcourſe of Combinations, Alterations, and Aliquot Parts*, bound up with his Algebra, page 109.

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ſents the numbers of quaternions, or quadruplets, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by four in a parcel, that occur in the ſaid twelve ſucceſſive lines: and the fifth vertical column, under the Roman numeral V, is the ſeries 0, 0, 0, 0, 1. 5, 15, 35, 70, 126, 252, 462, or the fifth order of figurate numbers (called the *triangulo-pyramidal*, or *trigono-pyramidal, numbers*) with four cyphers prefixed to them, and repreſents the numbers of quintuplets, or quinions, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by five in a parcel, that occur in the ſaid twelve ſucceſſive lines: and the ſixth vertical column, under the Roman numeral VI, is the ſeries 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, or the ſixth order of figurate numbers (called the *pyramido-pyramidal numbers*) with five cyphers prefixed to them, and repreſents the numbers of ſextuplets, or ſenions, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, or combinations of them by ſix in a parcel, that occur in the ſaid twelve ſucceſſive lines *. And, in like manner, the ſix following vertical columns, under the Roman numerals VII, VIII, IX, X, XI, and XII, contain the ſeventh, eighth, ninth, tenth, eleventh, and twelfth orders of figurate numbers, with ſix, ſeven, eight, nine, ten, and eleven, cyphers prefixed to them reſpectively, and repreſent the numbers of ſeptuplets,

* Dr. John Wallis, in his *Diſcourſe of Combinations, Alternations, and Aliquot Parts*, bound up with his Algebra, page 109, and Mr. Nicholas Mercator, in his *Logarithmotechnia*, publiſhed in the firſt volume of this collection of tracts, called *Scriptores Logarithmici*, page 178, call the 5th order of figurate numbers, to wit, 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, 330, &c, the *triangulo-triangular*, or *trigono-trigonal*, numbers, and the 6th order of figurate numbers, to wit, 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, 462, &c, the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, inſtead of calling the former the *triangulo-pyramidal*, or *trigono-pyramidal*, numbers, and the latter the *pyramido pyramidal* numbers, as our author calls them. So that there appears to be a variation amongſt different writers on this ſubject with reſpect to the names to be given to the figurate numbers of the 5th and 6th, and other higher orders. And therefore, to avoid ambiguity, it ſeems to be moſt convenient to denote the figurate numbers of the 5th and 6th, and all higher, orders, only by the numbers or exponents, of their orders, calling them the figurate numbers of the fifth, and the 6th, and the 7th, and the 8th, and other following higher orders.

octuplets, noncuplets, decuplets, undecuplets, and duodecuplets, or of septenions, octonions, novenions, denions, undenions, and duodenions, of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, and *m*, respectively, or combinations of them by seven in a parcel, and by eight in a parcel, and by nine in a parcel, and by ten in a parcel, and by eleven in a parcel, and by twelve in a parcel, that occur in the said twelve successive lines.

A Table of the first twelve orders, or serieses, of figurate numbers, or of all the different combinations that may be made of twelve different letters a, b, c, d, e, f, g, h, i, k, l, *and* m, *by taking them, first, singly, and then combining them with each other in parcels consisting of two letters, of three letters, of four letters, of five letters, of six letters, of seven letters, of eight letters, of nine letters, of ten letters, of eleven letters, and of twelve letters.*

The Roman numerals at the top of the table are the exponents of the combinations of the letters a, b, c, d, e, f, g, h, i, k, l, m, *exhibited by the several vertical columns in it.*

lett.	*num*	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
a.	1.	1	0	0	0	0	0	0	0	0	0	0	0
b.	2.	1	1	0	0	0	0	0	0	0	0	0	0
c.	3.	1	2	1	0	0	0	0	0	0	0	0	0
d.	4.	1	3	3	1	0	0	0	0	0	0	0	0
e.	5.	1	4	6	4	1	0	0	0	0	0	0	0
f.	6.	1	5	10	10	5	1	0	0	0	0	0	0
g.	7.	1	6	15	20	15	6	1	0	0	0	0	0
h.	8.	1	7	21	35	35	21	7	1	0	0	0	0
i.	9.	1	8	28	56	70	56	28	8	1	0	0	0
k.	10.	1	9	36	84	126	126	84	36	9	1	0	0
l.	11.	1	10	45	120	210	252	210	120	45	10	1	0
m.	12.	1	11	55	165	330	462	462	330	165	55	11	1

The

The firft vertical column on the left hand contains the letters that are to be combined together.

The fecond column expreffes the numbers of letters that are to be combined together.

50. The properties of the numbers exhibited in the foregoing table are truly curious and furprifing. For it not only contains in it (as we have feen in the foregoing pages) the clue to the myfterious doctrine of combinations, but it is alfo the ground, or foundation, of moft of the important and abftrufe difcoveries that have been made in the other branches of the mathematicks, as is well known to thofe perfons who are fkilled in the higher parts of geometry. We fhall here give a flight fketch, and but a flight one, of fome of the faid properties, without a formal demonftration of any of them except the twelfth and laft, which is that which is moft immediately connected with the fubject of combinations which we are here inquiring into; the other eleven properties being either eafy confequences of the faid 12th property, or being fufficiently evident from the manner in which the foregoing table was conftructed, or the feveral orders of figurate numbers were generated from each other.

Some wonderful properties of the foregoing table of combinations.

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51. The firft property. The fecond of the vertical columns of numbers in the faid table, or that which is placed under the exponent II, begins with one cypher; the third column begins with two cyphers; the fourth column with three cyphers; and, in general, every column with as many cyphers, wanting one, as there are units in the exponent of the combinations reprefented by it, fo that, if the exponent

 of

of the column is c, the number of cyphers in the beginning of the column will be $c-1$.

This property is too evident to need, or, perhaps, to admit of, any proof.

52. The second property. The first significant terms of the several vertical columns, taken in their order in a slanting line downwards from the top of the table on the left hand to the bottom of it on the right hand, are the same with the significant terms of the first vertical column; and the second significant terms of the several vertical columns, taken in the same manner, are the same with the significant terms of the second vertical column; and the third significant terms of the several vertical columns, taken in the same manner, are the same with the significant terms of the third vertical column; and, in like manner, the fourth, fifth, sixth, and other following, significant terms of the several vertical columns are the same with the significant terms of the fourth, fifth, sixth, and other following, vertical columns, respectively: so that the first of those oblique lines of terms constitutes a series of units, or 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, or the first order of figurate numbers; and the second of those oblique lines of terms constitutes a series of the natural, or lateral, numbers, or 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or the second order of figurate numbers; and the third of those oblique lines of terms constitutes a series of the trigonal or triangular numbers, to wit, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, or the third order of figurate numbers; and, in like manner, the fourth, fifth, sixth, and other following oblique lines of terms, constitute the fourth, fifth, sixth, and other following orders of figurate numbers respectively.

53. The third property. The second term in every vertical column, beginning from 1, is the same number as the exponent of the combinations exhibited by the said column, or as the number denoted by a Roman numeral at the top of the column, which denotes its place or order in the table. Thus, the second term of the fourth vertical column 1, 4, 10, 20, 35, 56, 84, 120, 165, reckoning from 1, is 4; and

and the ſecond term of the fifth vertical column is 5, and the ſecond term of the ſixth vertical column is 6; and the ſame thing may be obſerved in all the following columns.

This property is too evident to need a proof.

54. The fourth property. Every term in the foregoing table is equal to the ſum of all the terms that ſtand above it in the next preceeding vertical column. Thus, for example, 56 (which is the ſixth ſignificant term in the fourth vertical column) is equal to the ſum of 1, 3, 6, 10, 15, and 21, which are the firſt ſix ſignificant terms of the next preceeding, or third, vertical column, and which all ſtand above the ſaid term 56, or above the term 28 in the third vertical column, which ſtands even with the ſaid term 56 in the fourth column.

This property is manifeſt from art. 40, 41, 42, &c. — — 48.

55. The fifth property. Every term in the table, after the firſt term 1 in each horizontal row of terms, is equal to the ſum of the two terms that ſtand immediately above it in the ſame vertical column and in the next preceeding vertical column. Thus, for example, 56 (which is the 9th term in the fourth vertical column, including the cyphers, or the ſixth term excluſive of the cyphers) is equal to the ſum of 35, which is the term next above 56 in the ſame vertical column, and 21, which is the term next above 56 in the next preceeding or third vertical column. And, from the manner in which the table is formed, the ſame thing is evident of every other term in the table.

56. The ſixth property. The terms of every tranſverſe, or horizontal, column increaſe gradually from 1 to a certain magnitude, and then decreaſe again by the ſame degrees to 1, ſo as to make the terms that are equidiſtant from the beginning and the end of the column be equal to each other. Thus, for example, the terms of the 7th tranſverſe, or horizontal column are 1, 6, 15, 20, 15, 6, 1, in which the firſt and the laſt term are, both of them, 1, the ſecond

term

term and the laſt term but one, are, both of them, 6, the third term and the laſt term but two are, both of them, 15, and the middle term is 20, which is greater than any of the others. And the ſame thing may be obſerved in all the other tranſverſe or horizontal columns.

57. This property may be ſhewn to be general or to extend to all the tranſverſe, or horizontal columns, how numerous ſoever, to which we may ſuppoſe the table to be continued, as well as to the twelve columns ſet down in the foregoing table, by proving that, if it is true in any one horizontal column, (as we have ſeen that it is in all the horizontal columns ſet down in the foregoing table), it will alſo be true in the next following horizontal column. Now this may be proved in the manner following.

It appears from the fifth property already mentioned, that the ſecond and other following terms of every new tranſverſe, or horizontal, column are equal to the ſums of the two terms that ſtand immediately above them in the ſame vertical column, and in the next preceeding vertical column, reſpectively, or to the ſucceſſive ſums of the terms of the next preceeding horizontal column, taken two by two. Thus, for example, the ſecond, third, and other following terms of the 8th horizontal column are 7, 21, 35, 35, 21, 7, and 1; of which 7 is = 1 + 6, or the ſum of the firſt and ſecond term of the next preceeding, or 7th horizontal column 1, 6, 15, 20, 15, 6, 1; and 21 is = 6 + 15, or the ſum of the ſecond and third terms of the ſaid ſeventh horizontal column; and 35 is = 15 + 20, or the ſum of the third and fourth terms of the ſaid ſeventh horizontal column; and the ſecond 35 is = 20 + 15, or the ſum of the fourth and fifth terms of the ſaid ſeventh horizontal column; and the ſecond 21 is = 15 + 6, or the ſum of the fifth and ſixth terms of the ſaid ſeventh horizontal column; and the ſecond 7 is = 6 + 1, or the ſum of the ſixth and ſeventh terms of the ſaid ſeventh horizontal column; and the ſecond 1 is = 1 + 0, or the ſum of the ſeventh and eighth terms of the ſaid 7th horizontal column. So that the terms of the ſaid 8th horizontal column, 1, 7, 21, 35,

35,

35, 21, 7, 1, may be derived from the terms of the next preceeding, or 7th, horizontal column, 1, 6, 15, 20, 15, 6, 1, by ſetting down the latter terms twice following in two parallel horizontal rows, with the terms in the lower row advanced one ſtep to the right hand beyond thoſe in the upper row, ſo that the firſt term of the ſecond row ſhall be immediately under the ſecond term of the firſt row, and the ſecond term of the ſecond row under the third term of the firſt row, and the third term of the ſecond row under the fourth term of the firſt row, and every following term of the ſecond row under the next higher term of the firſt row, and then adding the correſpondent terms, or the terms which ſtand in the ſame vertical lines, together, in the manner following.

1, 6, 15, 20, 15, 6, 1
    1, 6, 15, 20, 15, 6, 1

1, 7, 21, 35, 35, 21, 7, 1

Now from this manner of deriving the terms of the 8th horizontal row of numbers from the 7th horizontal row of numbers, it is manifeſt that, ſince the terms of the ſeventh row that are equidiſtant from the two extreme terms 1 and 1, are equal to each other, the terms of the 8th row which are equidiſtant from the two extreme terms 1 and 1, muſt likewiſe be equal to each other, being the ſums of equal numbers that are added together in an oppoſite order, to wit, 6 + 1 and 1 + 6, 15 + 6 and 6 + 15, and 20 + 15 and 15 + 20. And this method of reaſoning will prove in like manner that, ſince the terms of the 8th horizontal row of numbers that are equidiſtant from the two extreme terms 1 and 1 are equal to each other, the terms of the 9th horizontal row of numbers that are equidiſtant from the two extreme terms 1 and 1 will alſo be equal to each other; and conſequently that, to whatever extent the table be ſuppoſed to be continued, the terms of every following horizontal row of numbers that are equidiſtant from the two extreme terms 1 and 1 will be equal to each other.

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58. The

58. The ſeventh property. If we take a certain number of vertical columns of numbers in the foregoing table, and continue the terms in each column till they are as many as there are columns, and then add up the ſeveral numbers in each column, and place the ſums thereby obtained in a new horizontal line, or row, at the bottom of the ſaid columns, the firſt of theſe ſums will be equal to the laſt but one, the ſecond of them to the laſt but two, the third of them to the laſt but three, and in general, the mth term to the laſt but m. Thus, for example, if we take the firſt eight vertical columns, and continue them to eight terms each (including the cyphers in the beginning of all but the firſt column), and then add the numbers in each ſeparate column into one ſum, the ſums thereby obtained will be 8, 28, 56, 70, 56, 28, 8, and 1; of which the firſt ſum 8 is equal to the laſt but one, the ſecond ſum 28 is equal to the laſt but two, and the third ſum 56 is equal to the laſt but three, and the fourth term 70 is itſelf the laſt term but four.

1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
1	2	1	0	0	0	0	0
1	3	3	1	0	0	0	0
1	4	6	4	1	0	0	0
1	5	10	10	5	1	0	0
1	6	15	20	15	6	1	0
1	7	21	35	35	21	7	1
8	28	56	70	56	28	8	1

59. This property follows from the fourth property, which is mentioned above in art. 54. For, by that property, each of the ſaid ſums is equal to the next following term of the next horizontal row of numbers, that is, in the example here given, of the ninth horizontal row of numbers in the foregoing table, which is 1, 8, 28, 56, 70, 56, 28, 8, and 1; to wit, the ſum of the units in the firſt vertical column is equal to the ſecond term 8 of the ſaid ninth horizontal row of numbers, and the ſum of the numbers 1, 2, 3, 4, 5, 6, 7, in the ſecond vertical column is equal to the third

8 term

term 28 of the ſaid ninth horizontal row of numbers; and the ſum of the numbers 1, 3, 6, 10, 15, 21, in the third vertical column is equal to the fourth term 56 of the ſaid ninth horizontal row of numbers; and in like manner, the ſums of the numbers 1, 4, 10, 20, 35, in the fourth vertical column, and of the numbers 1, 5, 15, 35, in the fifth vertical column, and of the numbers in the ſixth, ſeventh, and eighth vertical columns, are reſpectively equal to the numbers 70, 56, 28, 8, and 1, or the fifth, ſixth, ſeventh, eighth, and ninth, terms of the ſaid ninth horizontal row of numbers. But, by the ſixth property (which has been mentioned above in art. 56, and demonſtrated in art. 57) the terms of the ſaid ninth horizontal row of numbers, 1, 8, 28, 56, 70, 56, 28, 8, and 1, that are equidiſtant from the two extreme terms 1 and 1, are equal to each other, and muſt be ſo from the manner in which they are generated. Therefore the ſaid ſums of the numbers contained in the ſaid eight vertical columns, being equal to the ſecond, third, fourth, and other following terms of the ſaid ninth horizontal row of numbers, muſt be ſuch that, if an unit be prefixed to them (whereby their number will be increaſed to nine terms), the terms that are equidiſtant from the two extreme terms 1 and 1 will be equal to each other. And conſequently, if an unit be not prefixed to them, the firſt of thoſe ſums will be equal to the laſt of them but one, and the ſecond of them will be equal to the laſt but two, and the third of them will be equal to the laſt but three, and the mth of them will be equal to the laſt but m. And this, it is evident, will be true, if inſtead of eight vertical columns continued to eight terms each, we were to take any other number of vertical columns, how great ſoever, and continue them till the number of the terms in each column (including the cyphers) was equal to the number of the columns. And therefore it is true univerſally. Q. E. D.

60. The eighth property. The horizontal rows of numbers in the foregoing table of combinations, beginning with the ſecond row, exhibit the co-efficients of the ſeveral ſucceſſive powers of a binomial quantity, as $a + b$. Thus, the

numbers in the fecond horizontal row, to wit, 1 and 1, are the co-efficients of the two members a and b of the faid binomial quantity $a + b$ itfelf, or (as it is fometimes called) of the firft, or fimple, power of the faid binomial quantity. The numbers in the third horizontal row, to wit, 1, 2, and 1, are the co-efficients of the feveral terms aa, $2ab$, and bb, of the compound quantity $aa + 2ab + bb$, which is the fquare, or fecond power, of the faid binomial quantity. The numbers in the 4th horizontal row, to wit, 1, 3, 3, and 1, are the co-efficients of the feveral terms of the compound quantity $a^3 + 3a^2b + 3ab^2 + b^3$, which is the cube, or third power, of the faid binomial quantity. The numbers in the 5th horizontal row, to wit, 1, 4, 6, 4, and 1, are the co-efficients of the feveral terms of the compound quantity $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, which is the 4th power of the faid binomial quantity. And, in like manner, the numbers in the 6th, 7th, 8th, 9th, and every following horizontal row to the nth row (n being any whole number whatfoever) will be the co-efficients of the terms of the 5th, 6th, 7th, 8th, and every following power of the faid binomial quantity $a + b$, to the $\overline{n-1}|$th power, refpectively; and the numbers in the $\overline{n+1}|$th horizontal row of terms in the faid table will be the co-efficients of the terms of the nth power of the faid binomial quantity.

61. This property will appear from the manner in which the powers of the binomial quantity $a + b$ are generated from each other by multiplication, which is as follows:

$$
\begin{array}{l}
a+b \\
a+b \\
\hline
aa+ab \\
\quad +ab \quad +bb \\
\hline
aa+2ab \quad +bb = \overline{a+b}|^2 \\
\qquad\qquad a+b \\
\hline
a^3+2a^2b+abb \\
\quad + \ a^2b+2abb \quad +b^3 \\
\hline
a^3+3a^2b+3ab^2 \quad +b^3 = \overline{a+b}|^3 \\
\qquad\qquad\qquad a+b \\
\hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\quad + \ a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
a^4+4a^3b+6a^2b^2+4ab^3+b^4 = \overline{a+b}|^4.
\end{array}
$$

Or, if, for brevity's sake, we substitute $1+1$ instead of $a+b$, the multiplication will be as follows:

$$
\begin{array}{l}
1+1 \\
1+1 \\
\hline
1+1 \\
\quad +1+1 \\
\hline
1+2+1 = \overline{1+1}|^2 \\
\quad 1+1 \\
\hline
1+2+1 \\
\quad +1+2+1 \\
\hline
1+3+3+1 = \overline{1+1}|^3 \\
\qquad 1+1 \\
\hline
1+3+3+1 \\
\quad +1+3+3+1 \\
\hline
1+4+6+4+1 = \overline{1+1}|^4.
\end{array}
$$

Here

Here we ſee that every new power of the binomial quantity 1 + 1 is formed by adding together the terms of the next preceeding power of it by two at a time, or in ſetting down the terms of the preceeding power twice following in two parallel horizontal rows, with the terms in the lower row advanced one ſtep to the right hand beyond thoſe in the upper row; which is the manner in which the ſeveral horizontal rows of numbers in the foregoing table of combinations are, or may be, derived from each other, as appears from the 5th and 6th properties above-mentioned in art. 55, 56, and 57. Therefore, ſince theſe co-efficients of the terms of the powers of the binomial quantity 1 + 1 are generated from 1 + 1, in the ſame manner as the 3d, 4th, 5th, and other following rows of numbers in the foregoing table of combinations are generated from the ſame numbers 1 and 1 in the ſecond horizontal row, it follows that the numbers contained in the 3d, 4th, 5th, and other ſubſequent horizontal rows muſt coincide with, or be the ſame with, the co-efficients of the terms of the ſquare, cube, fourth power, and other following correſpondent powers of the ſaid binomial quantity. Q. E. D.

62. The ninth property. The ſums of the numbers contained in the ſeveral ſucceſſive horizontal rows in the foregoing table of combinations increaſe continually in the proportion of 1 to 2; or the ſum of the numbers in every new horizontal row is double of the ſum of the numbers in the next preceeding horizontal row. Theſe ſums are as follows:

In the 1ſt horizontal row 1 + 0 + 0 &c are = 1.
In the 2d horizontal row 1 + 1 + 0 + 0 &c are = 2.
In the 3d row 1 + 2 + 1 + 0 + 0 &c are = 4.
In the 4th row 1 + 3 + 3 + 1 + 0 + 0 &c are = 8.
In the 5th row 1 + 4 + 6 + 4 + 1 + 0 + 0 &c are = 16.
In the 6th row 1 + 5 + 10 + 10 + 5 + 1 &c are = 32.
In the 7th row 1 + 6 + 15 + 20 + 15 + 6 + 1 are = 64.
In the 8th row 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 are = 128.

Each

Each of these sums is double of the sum immediately preceeding it. And the same thing is true of the sums of the terms of the four following horizontal rows in the foregoing table, and of the sums of the terms of all the following horizontal rows that would belong to it, if it were continued to any greater number of vertical columns and horizontal rows whatsoever.

63. This property follows from what has been shewn above in art. 57, to wit, that every new horizontal row of numbers in the said table may he derived from the next preceeding horizontal row of numbers by setting down the numbers of the said preceeding row twice following in two parallel horizontal rows, with the terms in the lower row advanced one step further to the right hand than the terms in the upper row, and then adding the terms of the two rows that stand in the same vertical lines together. For the sum of the numbers contained in the new horizontal line arising from the addition of the said two lines together, must evidently be double of the sum of the numbers in only one of the lines so added. Thus, if we set down the numbers of the 7th horizontal row of numbers, to wit, 1, 6, 15, 20, 15, 6, 1, twice following in two parallel rows one under the other, as follows,

1, 6, 15, 20, 15, 6, 1
    1, 6, 15, 20, 15, 6, 1,

and then add the two rows together, so as to make a new line of numbers, to wit,

1, 7, 21, 35, 35, 21, 7, 1,

it is evident that the sum of the numbers contained in this new line must be double of the sum of the numbers contained in either of the two former lines. Q. E. D.

64. This property may also be derived from the last or 8th property, set forth and proved in art. 60 and 61. For, since every new power of the binomial quantity 1 + 1 must be greater than the next preceeding power of it in the proportion of 1 + 1, or 2, to 1; and it has been shewn in art. 61, that the several horizontal rows of numbers in the fore-

5 going

going table exhibit the members of the ſeveral ſucceſſive powers of the binomial quantity $1+1$; it follows that the ſaid horizontal rows of numbers muſt be greater, one than the other, in the ſame proportion of $1+1$, or 2, to 1.

Q. E. D.

65. The tenth property. If the ſums of the numbers contained in the ſeveral horizontal rows of the foregoing table of combinations be continually added to each other, the new ſums thence ariſing, or the ſums of the former ſums, will form a ſeries of numbers, which will be equal to the ſeveral powers of 2, with an unit ſubtracted from them, or to $2-1$, $\overline{2}|^2-1$, $\overline{2}|^3-1$, $\overline{2}|^4-1$, $\overline{2}|^5-1$, $\overline{2}|^6-1$, $\overline{2}|^7-1$, &c, or to $2-1$, $4-1$, $8-1$, $16-1$, $32-1$, $64-1$, $128-1$, &c, or 1, 3, 7, 15, 31, 63, 127, &c.

Thus, for example, the ſums of the eight firſt horizontal rows of numbers are 1, 2, 4, 8, 16, 32, 64, and 128, reſpectively, as we have ſeen in art. 62. Now, if theſe ſums are added together, we ſhall have

$1=1=2-1$

$1+2=3=4-1=\overline{2}|^2-1$,

$1+2+4=7=8-1=\overline{2}|^3-1$,

$1+2+4+8=15=16-1=\overline{2}|^4-1$,

$1+2+4+8+16=31=32-1=\overline{2}|^5-1$,

$1+2+4+8+16+32=63=64-1=\overline{2}|^6-1$,

$1+2+4+8+16+32+64=127=128-1=\overline{2}|^7-1$, and

$1+2+4+8+16+32+64+128=255=256-1=\overline{2}|^8-1$.

And univerſally, if the number of the horizontal rows be n, the ſum of the ſums of all the numbers contained in them will be $\overline{2}|^n-1$.

66. This is evident from art. 33. For the ſum of all theſe ſums is the number of all the poſſible combinations of n letters; which is ſhewn in art. 33 to be $=\overline{2}|^n-1$. Therefore the ſucceſſive ſums of the ſums of all the numbers contained

tained in the said horizontal rows will form the series 1, $2-1$, $\overline{2}|^2-1$, $\overline{2}|^3-1$, $\overline{2}|^4-1$, $\overline{2}|^5-1$, $\overline{2}|^6-1$, $\overline{2}|^7-1$, $\overline{2}|^8-1$, &c .... $\overline{2}|^{n-1}$. Q. E. D.

67. The eleventh property. If we divide the terms of the second, or any other of the vertical columns of the foregoing table of combinations by the corresponding terms (or terms situated in the same horizontal line) of the next preceeding vertical column, the several quotients thence arising will be equal to the terms of an arithmetical progression consisting of fractions, of which the common difference is a fraction of which an unit is the numerator, and the number which is the exponent of the first of the said vertical columns (by the terms of which the terms of the other vertical column are divided) is the denominator.

Thus, for example, if we divide each of the terms of the second vertical column 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 (omitting the first term, which is a cypher, or 0), by the corresponding term of the first vertical column 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, the quotients will be $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, $\frac{4}{1}$, $\frac{5}{1}$, $\frac{6}{1}$, $\frac{7}{1}$, $\frac{8}{1}$, $\frac{9}{1}$, $\frac{10}{1}$, $\frac{11}{1}$, which differ from each other by the common difference $\frac{1}{1}$.

And, if we divide each of the terms of the third vertical column, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55 (omitting the two cyphers) by the corresponding term of the second vertical column, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, the quotients will be $\frac{1}{2}$, $\frac{3}{3}$, $\frac{6}{4}$, $\frac{10}{5}$, $\frac{15}{6}$, $\frac{21}{7}$, $\frac{28}{8}$, $\frac{36}{9}$, $\frac{45}{10}$, and $\frac{55}{11}$, which are respectively equal to $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, $\frac{6}{2}$, $\frac{7}{2}$, $\frac{8}{2}$, $\frac{9}{2}$, and $\frac{10}{2}$, which differ from each by the fraction $\frac{1}{2}$, of which 1 is the numerator, and 2, or the exponent of the second vertical column 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 11, (by the terms of which the terms of the third vertical column are divided), is the denominator.

If we divide the terms of the fourth vertical column (omitting the cyphers at the beginning) by the corresponding terms of the third column, the quotients will be as follows; to wit,

$$\frac{1}{3} = \frac{1}{3}$$
$$\frac{4}{6} = \frac{2}{3}$$
$$\frac{10}{10} = \frac{3}{3}$$
$$\frac{20}{15} = \frac{4}{3}$$
$$\frac{35}{21} = \frac{5}{3}$$
$$\frac{56}{28} = \frac{6}{3}$$
$$\frac{84}{36} = \frac{7}{3}$$
$$\frac{120}{45} = \frac{8}{3}$$

and $\frac{165}{55} = \frac{9}{3}$, which differ from each other by the fraction $\frac{1}{3}$, of which 1 is the numerator, and 3, or the exponent of the third vertical column (by the terms of which the terms of the fourth vertical column are divided), is the denominator.

In like manner the quotients that arise by dividing the terms of the fifth vertical column by the corresponding terms of the fourth vertical column are the following, to wit, $\frac{1}{4}$, $\frac{5}{10}$, $\frac{15}{20}$, $\frac{35}{35}$, $\frac{70}{56}$, $\frac{126}{84}$, $\frac{210}{120}$, and $\frac{330}{165}$, which are respectively equal to $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{5}{4}$, $\frac{6}{4}$, $\frac{7}{4}$, and $\frac{8}{4}$.

And the quotients that arise by dividing the terms of the sixth vertical column by the corresponding terms of the fifth

fifth vertical column are $\frac{1}{5}$, $\frac{6}{15}$, $\frac{21}{35}$, $\frac{56}{70}$, $\frac{126}{126}$, $\frac{252}{210}$, and $\frac{462}{330}$, which are refpectively equal to $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{5}$, $\frac{6}{5}$, and $\frac{7}{5}$.

The quotients that arife by dividing the terms of the feventh vertical column by the correfponding terms of the fixth column are $\frac{1}{6}$, $\frac{7}{21}$, $\frac{28}{56}$, $\frac{84}{126}$, $\frac{210}{252}$, and $\frac{462}{462}$, which are refpectively equal to $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, and $\frac{6}{6}$.

The quotients that arife by dividing the terms of the eighth vertical column by the correfpondent terms of the feventh column are $\frac{1}{7}$, $\frac{8}{28}$, $\frac{36}{84}$, $\frac{120}{210}$, and $\frac{330}{462}$, which are refpectively equal to $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, and $\frac{5}{7}$.

The quotients that arife by dividing the terms of the ninth vertical column by the correfponding terms of the eighth column are $\frac{1}{8}$, $\frac{9}{36}$, $\frac{45}{120}$, and $\frac{165}{330}$, which are refpectively equal to $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, and $\frac{4}{8}$.

The quotients that arife by dividing the terms of the tenth vertical column by the correfpondent terms of the ninth column are $\frac{1}{9}$, $\frac{10}{45}$, $\frac{55}{165}$, which are refpectively equal to $\frac{1}{9}$, $\frac{2}{9}$, and $\frac{3}{9}$.

And the quotients that arife by dividing the terms of the eleventh vertical column by the correfpondent terms of the tenth column are $\frac{1}{10}$ and $\frac{11}{55}$, which are refpectively equal to $\frac{1}{10}$ and $\frac{2}{10}$.

68. This property of thefe numbers might, if it were neceffary to the main object of this Treatife, be derived from

 the

the following, or 12th, property of the ſaid numbers, which we will now proceed to ſet forth and to demonſtrate.

69. The twelfth property. The ſum of all the numbers contained in any one of the vertical columns of the foregoing table of combinations, is to the ſum of the like number of terms that ſhould be all equal to the laſt term of the column, in the ſame proportion as 1 to the exponent of the column, or the number which denotes the place of the column, and which is marked by a Roman numeral figure at the top of it.

Thus, in column 1ſt, which conſiſts entirely of units, the ſum of all the twelve terms is 12, which is to the ſum of twelve terms all equal to the laſt term as 1 to 1, or the exponent of the firſt column. This propoſition is ſelf-evident.

In the 2d column the twelve terms are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11; the ſum of which is 66. And the ſum of twelve terms equal to the laſt, or greateſt, term 11, is 132. Now 66 is to 132 as 1 is to 2, or the exponent of this ſecond column.

In the 3d column the twelve terms are 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, and 55; the ſum of which is 220. And the ſum of twelve terms equal to the laſt, or greateſt, term 55, is 660. Now 220 is to 660 as 1 is to 3, or the exponent of the third column.

In the 4th column the twelve terms are 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, 120, and 165; the ſum of which is 495. And the ſum of twelve terms equal to the laſt, or greateſt, term 165, is 1980. Now 495 is to 1980 as 1 is to 4, or the exponent of the fourth column.

In the 5th column the twelve terms are 0, 0, 0, 0, 1, 5, 15, 35, 70, 126, 210, and 330; the ſum of which is 792. And the ſum of twelve terms equal to the laſt, or greateſt, term 330, is 3960. Now 792 is to 3960 as 1 is to 5, or the exponent of the fifth column.

In the 6th column the twelve terms are 0, 0, 0, 0, 0, 1, 6, 21, 56, 126, 252, and 462; the ſum of which is 924. And

And the ſum of twelve terms equal to the laſt, or greateſt, term 462, is 5544. Now 924 is to 5544 as 1 is to 6, or the exponent of the ſixth column.

In the 7th column the twelve terms are 0, 0, 0, 0, 0, 0, 1, 7, 28, 84, 210, and 462; the ſum of which is 792. And the ſum of twelve terms equal to the laſt, or greateſt, term 462, is 5544. Now 792 is to 5544 as 1 is to 7, or the exponent of the ſeventh column.

In the 8th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 1, 8, 36, 120, and 330; the ſum of which is 495. And the ſum of twelve terms equal to the laſt, or greateſt, term 330, is 3960. Now 495 is to 3960 as 1 is to 8, or the exponent of the eighth column.

In the 9th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 1, 9, 45, and 165; the ſum of which is 220. And the ſum of twelve terms equal to the laſt, or greateſt, term 165, is 1980. Now 220 is to 1980 as 1 is to 9, or the exponent of the ſaid ninth column.

In the 10th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 10, and 55; the ſum of which is 66. And the ſum of twelve terms all equal to the laſt, or greateſt, term 55, is 660. Now 66 is to 660 as 1 is to 10, or the exponent of the ſaid tenth column.

In the 11th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, and 11; the ſum of which is 12. And the ſum of twelve terms all equal to the laſt, or greateſt, term 11, is 132. Now 12 is to 132 as 1 is to 11, or the exponent of the ſaid eleventh column.

In the 12th column the twelve terms are 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, and 1; the ſum of which is 1. And the ſum of twelve terms all equal to the laſt, or greateſt, term 1, is 12. Now 1 is to 12 as 1 is to the exponent of the ſaid twelfth column, that exponent being 12.

And the ſame thing will be found to be true, if, inſtead of taking twelve terms in each of the ſaid vertical columns, we were to take any leſſer number, as five, or ſix, or ſeven, terms, or any greater number of terms whatſoever, as fifteen, or twenty, or a hundred terms, continuing the table both downwards and sideways for that purpose, namely, that, if the exponent of any one of the vertical columns be called *c*, the sum of all the terms in the said column, continued to any number of terms whatsoever, will be to the sum of as many terms all equal to the last, or greatest, term of the said column, as 1 is to *c* *.

70. This

* The attentive reader may perhaps have observed, in reading the foregoing translation of the twelve surprising properties of the numbers contained in the table of combinations, exhibited in page 74, (which properties are set forth in pages 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, and 91) that the 7th property (which is set forth in art. 58, page 80) is not the same with the 7th property in the author's original, (which is contained in page 19), but answers to the 8th property set forth in the said original; and that there is no property in the translation that exactly answers to the said 7th property in the original. The reason of this omission is, that the 7th property set forth in the author's original, seemed to me to be the same with the next preceeding, or 6th, property, and therefore to be an unnecessary repetition. But, whether it is so, or not, must be referred to the reader's judgment. And therefore I will here set down both the 6th and the 7th property, as they are expressed in the original.

The sixth property is expressed in these words. *Columnæ cujusvis transversæ termini ab unitate aliquousque crescunt, deinde per eosdem gradus rursùm decrescunt. Idem intellige de summis columnarum verticalium æquè-altarum, seu terminis sequentis columnæ transversæ, per quartam proprietatem.*

And the seventh property is expressed in these words. *Columnarum verticalium æquè-altarum bases, sive termini columnæ transversæ cujuslibet, primus quidem et ultimus significativus perpetuò inter se æquantur, ut et secundus et penultimus, tertius et antepenultimus, atque ita porrò, si columna pluribus terminis significativis constet.*

Now this 7th property seems to me to be a mere repetition of the foregoing 6th property, and particularly of the first sentence of it, to wit, *Columnæ cujusvis transversæ termini ab unitate aliquousque crescunt, deinde per eosdem gradus rursùm decrescunt.* These words, "*ab unitate crescunt, deinde per eosdem gradus rursùm decrescunt*" seem only to be paraphrased, or more fully explained, by the words of the 7th property, to wit, *primus quidem et ultimus perpetuò inter se æquantur, ut et secundus et penultimus, tertius et antepenultimus, atque ita porrò.* Therefore, as I could find no new meaning to the words of the 7th property, whereby it could be distinguished from the 6th property, I thought it better to omit it.

Yet we may observe that there are twelve properties of the figurate numbers, or numbers contained in the foregoing table of combinations, set down in the translation as well as in the author's original. This is owing to my having divided the 10th property of these numbers mentioned in the author's original

70. This is the moſt important property belonging to the figurate numbers, and that which will be of moſt uſe to us in treating of the doctrine of chances, or the art of forming probable conjectures concerning events that depend on chance, which is the ſubject of this treatiſe. I ſhall therefore now endeavour to demonſtrate this property of the ſaid numbers in a ſcientific and ſatisfactory manner, and ſo as to convince my readers that it muſt be true in all caſes whatſoever (however great the number of vertical columns, and that of the terms in each vertical column, may be ſuppoſed to be taken) as well as in the ſmall number of caſes exhibited in the foregoing table of combinations. And in order to this I ſhall proceed to lay down the four following preliminary propoſitions, or lemmas, as the ground-work of the following demonſtration.

Lemma I.

71. The ſum of any number of terms whatſoever in the firſt vertical column in the foregoing table of combinations, is equal to the ſum of an equal number of terms that are all equal to the laſt term.

original (which conſiſts of two branches) into two ſeparate properties, calling the firſt branch of the ſaid 10th property in the original (which is expreſſed in theſe words, "*Summæ ſerierum tranſverſarum progrediuntur in continuâ ratione duplâ*") the 9th property in the tranſlation; and the ſecond branch of it (which is expreſſed in theſe words, "*Summarum verò ſummæ, ab initio collectæ, terminos conſtituunt progreſſionis duplæ unitate multatos*") the 10th property in the tranſlation.

The differences therefore between the properties of theſe numbers, as expreſſed in the tranſlation, and as expreſſed in the original, are as follows. The ſix firſt properties of theſe numbers in the tranſlation anſwer to the ſix firſt properties of them in the original reſpectively; the 7th property in the tranſlation anſwers to the 8th property in the original: the 8th property in the tranſlation anſwers to the 9th property in the original: the 9th and 10th properties in the tranſlation anſwer to the firſt and ſecond branches of the 10th property in the original: and the 11th and 12th properties in the tranſlation anſwer to the 11th and 12th properties in the original, reſpectively.

Demon-

Demonstration.

This is evident, becaufe all the terms in the firft column are units, or equal to the laft term. Therefore the fum of all the faid terms is the fum of the fame number of terms equal to the laft term. Q. E. D.

Lemma II.

72. If in any one of the vertical columns of numbers in the foregoing table of combinations, after the firft column, we take as many terms (including the cyphers in the beginning of the column) as there are units in the exponent of the column, the fum of all the faid terms will be to the fum of the fame number of terms that are all equal to the laft of them in the fame proportion as 1 is to the exponent of the faid column.

Demonstration.

By the firft property of thefe figurate numbers, fet forth in art. 51, the number of cyphers at the beginning of each of the faid vertical columns is lefs by an unit than the exponent of the faid column. And, by the fecond property of thefe numbers, fet forth in art. 52, the firft term in every column after the cyphers is an unit. Therefore the fum of all the terms of the vertical column that are fuppofed in this lemma to be taken (which are only as many as there are units in the exponent of the column) will be the fum of fome cyphers and an unit, and confequently will be equal only to an unit. And the fum of the fame number of terms all equal to the laft will be equal to the fum of the fame number of terms all equal to an unit, or will be equal to the exponent of the column. Therefore the fum of all the terms in the faid vertical column will be to the fum of as many

terms

terms all equal to the laſt term (which is an unit) in the ſame proportion as 1 is to the exponent of the column.

Q. E. D.

Thus, for example, in the 4th vertical column, if we take only the four firſt terms 0, 0, 0, 1, the ſum of theſe terms will be $= 1$, and the ſum of four terms all equal to the laſt term, which is 1, will be $(= 1 + 1 + 1 + 1) = 4$. And therefore the former ſum is to the latter as 1 is to 4, which is the exponent of the ſaid column. And, in general, if the exponent of the column be c, and we take c terms in it, the $c - 1$ terms will be all cyphers, and the cth term will be 1. Therefore the ſum of the ſaid c terms will be 1. And the ſum of c terms all equal to the laſt term (which is 1) will be c. Therefore the former ſum will be to the latter ſum as 1 is to c.

This lemma is the ſame with the general propoſition hereafter to be proved, or the 12th property of the figurate numbers, in the caſe of taking only the firſt ſignificant term in each of the vertical columns, which firſt term is always an unit.

Lemma III.

73. If the above deſcribed 12th property of the figurate numbers (which we are preparing to demonſtrate the truth of) ſhould be found to be true in any one of the vertical columns of numbers contained in the foregoing table of combinations, or the ſum of any number of terms taken in the ſaid column ſhould be to the ſum of the ſame number of terms all equal to the laſt, or greateſt, term, always in the ſame conſtant proportion, whatever be the number of terms ſo taken; and this proportion be that of 1 to a certain number denoted by r, ſo that the ſum of the terms ſo taken ſhall always be equal to the quotient that ariſes by dividing the latter ſum (or the ſum of the ſame number of terms all equal to the laſt term) by the number r; the exceſs of

of the number of the terms ſo taken above the number r will be to the exceſs of the number of terms ſo taken above 1 in the ſame proportion as the laſt term but one of the terms ſo taken to the laſt term of all.

Demonstration.

Let the terms ſuppoſed to be taken in the ſaid vertical column be A, B, C, D, &c . . . . K, L, of which L is the laſt, and K the laſt but one. And let the number of the terms ſo taken be n. We are then to prove that $n-r$ will be to $n-1$ as K is to L.

Now, ſince the ſum of all the n terms A, B, C, D, &c . . . . K, L is ſuppoſed to be to the ſum of n terms all equal to the laſt term L, or to $n \times L$, in the proportion of 1 to r; and the ſum of all the terms A, B, C, D, &c . . . . K, L, except the laſt term L, or the ſum of all the $n-1$ terms A, B, C, D, &c . . . . K, is alſo ſuppoſed to be to the ſum of $n-1$ terms all equal to the laſt term K, or to $\overline{n-1} \times K$, in the ſame proportion of 1 to r; it follows that the ſum of all the n terms A, B, C, D, &c . . . . K, L will be $= \frac{n \times L}{r}$, and the ſum of all the $n-1$ terms A, B, C, D, &c, K, will be $= \frac{\overline{n-1} \times K}{r}$. But this latter ſum, or $A + B + C + D +$ &c $+ K$, is leſs than the former ſum, or $A + B + C + D +$ &c $K + L$, by L. Therefore $\frac{\overline{n-1} \times K}{r}$ is $= \frac{n \times L}{r} - L = \frac{n \times L - r \times L}{r} = \frac{\overline{n-r} \times L}{r}$, and conſequently $\overline{n-1} \times K$ is $= \overline{n-r} \times L$. Therefore $n-r$ is to $n-1$ as K is to L.

Q. E. D.

LEMMA

Lemma IV.

74. If in the foregoing table of combinations, or figurate numbers, we take two contiguous vertical columns; and the numbers in the firſt of the two columns are found to have the twelfth property above-deſcribed, or the ſum of any number of terms of it is to the ſum of as many terms all equal to the laſt, or greateſt, term, in the ſame proportion as the ſum of any other number of its terms is to the ſum of as many terms all equal to the laſt, or greateſt, of this latter number of terms; and the ſaid proportion is that of 1 to the number r; and in the ſecond of the ſaid two contiguous vertical columns it be found that for a certain number of terms the numbers in the ſaid column are likewiſe poſſeſſed of the ſame 12th property, and that the ſum of the ſaid number of terms is to the ſum of as many terms all equal to the laſt, or greateſt, term in the proportion of 1 to $r+1$, and that the ſum of any leſſer number of its terms is to the ſum of as many terms all equal to the laſt, or greateſt, of the ſaid leſſer number of terms, in the ſame proportion of 1 to $r+1$; I ſay, then, that, if we take another term in the ſaid ſecond vertical column above the number before taken, and in which the ſaid 12th property has been found to take place, the ſaid 12th property will take place likewiſe with reſpect to the numbers in the ſaid ſecond column, when increaſed by the ſaid new term, and the ſum of all the terms in the ſaid column, including the ſaid new term, will be to the ſum of as many terms all equal to the ſaid new term, in the ſame proportion of 1 to $r+1$.

Demonstration.

Let n be the number of terms that are taken in the ſecond of the two vertical columns; and let the ſame number of terms be taken in the firſt of them. Let the terms in the ſaid firſt column be A, B, C, D, &c .... K, L, and thoſe in the ſecond column be a, b, c, d, &c ..., k, l. Then, by the ſuppoſition, the ſum of the n terms A, B, C, D, &c

 ....K,

. . . . K, L, will be to the sum of n terms, all equal to the last term L, or to $n \times$ L, in the proportion of 1 to r; and the sum of all the n terms a, b, c, d, *&c* . . . . k, l, will be to the sum of n terms all equal to l, or to $n \times l$, in the proportion of 1 to $r+1$. Now let another term m be added to the former terms a, b, c, d, *&c* . . . . k, l, of the second of the said two vertical columns. We are then to prove that the sum of all the terms a, b, c, d, *&c* . . . . k, l, and m (the number of which is $n+1$) will be to the sum of as many terms all equal to the last term m, or to $\overline{n+1} \times m$, in the same proportion of 1 to $r+1$.

Now, by the 4th property of the figurate numbers above set forth in art. 54, it is manifest that l, or the nth term of the second of the two vertical columns, will be equal to the sum of all the terms in the preceeding vertical column except the last term L, or to the sum of the $n-1$ terms A, B, C, D, &c . . . . K. But, by the supposition, the sum of these terms is less than the sum of as many terms equal to the last term K in the proportion of 1 to r, or is equal to $\frac{\overline{n-1} \times K}{r}$. Therefore l is $= \frac{\overline{n-1} \times K}{r}$.

But, because the above-described 12th property is supposed to belong to the numbers of the first of the said two vertical columns, to wit, A, B, C, D, &c . . . . K, L, and the sum of any number of terms in the said column is supposed to be to the sum of as many terms all equal to its last term in the constant proportion of 1 to r, it follows from lemma 3d, art. 73, that $n-r$ will be to $n-1$ as K is to L. Therefore $\overline{n-1} \times$ K will be $= \overline{n-r} \times$ L; and consequently $\frac{\overline{n-1} \times K}{r}$ will be $= \frac{\overline{n-r} \times L}{r}$.

Therefore l (which has been shewn to be $= \frac{\overline{n-1} \times K}{r}$) will be $= \frac{\overline{n-r} \times L}{r}$; and consequently $n-r$ will be to l as r is to L.

But, by the supposition, the sum of the terms A, B, C, D, &c . . . K, L, is to the sum of the same number of terms all

all equal to the laſt term L, or to $n \times L$, in the proportion of 1 to r. Therefore $r \times \overline{A + B + C + D + \&c + K + L}$ is $= n \times L$; and conſequently r is to L as n is to $A + B + C + D + \&c + K + L$.

Therefore $n - r$ will be to l as n is to $A + B + C + D + \&c + K + L$.

But, by the 4th property of the figurate numbers above ſet forth in art. 54, m (which is the $n + 1$th term of the ſecond vertical column) is equal to the ſum of A, B, C, D, &c, K, L, or the n firſt terms of the preceeding column. Therefore $n - r$ will be to l as n is to m; and conſequently $\overline{n - r} \times m$ will be $= n \times l$.

But, by the ſuppoſition concerning the numbers in the ſecond vertical column, the ſum of the firſt n terms of it, to wit, $a + b + c + d + \&c + k + l$, is to the ſum of as many terms all equal to the laſt term l, or to $n \times l$, in the proportion of 1 to $r + 1$. Therefore $\overline{r + 1} \times \overline{a + b + c + d + \&c + k + l}$ is $= n \times l$; and conſequently $\overline{n - r} \times m$ (which is equal to $n \times l$) will be $= \overline{r + 1} \times \overline{a + b + c + d + \&c + k + l}$. Therefore $n - r : r + 1 :: a + b + c + d + \&c + k + l : m$. Therefore, *componendo*, we ſhall have ($n - r + r + 1$, or) $n + 1 : r + 1 :: a + b + c + d + \&c + k + l + m : m$; and *permutando*, $n + 1 : a + b + c + d + \&c + k + l + m :: r + 1 : m$; and, *invertendo*, $a + b + c + d + \&c + k + l + m : n + 1 :: m : r + 1$. But $n + 1$ is to $\overline{n + 1} \times m :: \overline{r + 1} : \overline{r + 1} \times m$. Therefore, *ex æquo*, $a + b + c + d + \&c + k + l + m : \overline{n + 1} \times m :: m : \overline{r + 1} \times m :: 1 : r + 1$; that is, the ſum of the firſt $n + 1$ terms of the ſecond vertical column will be to $n + 1$ times the laſt, or $\overline{n + 1}$th, term, m, of the ſaid column in the ſame proportion of 1 to $r + 1$ in which the ſum of the firſt n terms of it was to n times the laſt, or nth, term. Q. E. D.

Coroll. It follows from this lemma, that, if the number of terms in the ſecond of the two vertical columns be increased from n terms to any other number of terms whatsoever denoted by $n + p$, it will be true with respect to the column, when so increased, that the sum of all its terms will be to $n + p$ times the last, or greatest, term of it in the same proportion of 1 to $r + 1$. For the lemma may be successively extended from a column consisting of $n + 1$ terms, to a column consisting of $n + 2$ terms, and to a column consisting of $n + 3$ terms, and to a column consisting of $n + 4$ terms, and so on till we come to the column of $n + p$ terms; the reasonings being exactly the same in this extension of it to these several columns of $n + 2$ terms, $n + 3$ terms, $n + 4$ terms, &c, as in the lemma itself, in which, upon a supposition that the sum of n terms of the column is to n times the last, or greatest, or nth, term of it in the proportion of 1 to $r + 1$, it is shewn that the sum of $n + 1$ terms of it will be to $n + 1$ times the last, or greatest, or $\overline{n+1}$th, term of it in the same proportion of 1 to $r + 1$.

A demonstration of the 2d, 3d, and 4th foregoing Lemmas, contained in art. 72, 73, *and* 74, *by Mr. John Bernoulli, the author's brother.*

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75. Many years ago, when I communicated the foregoing propositions concerning the figurate numbers to my brother, Mr. John Bernoulli, he observed to me that the demonstrations of them might be made shorter and more elegant by uniting the three last of the four preceeding lemmas into one, in the manner following.

A LEMMA.

A Lemma.

If in a table of the figurate numbers (ſuch as the foregoing table of combinations, in page 71, art. 49), it be the property of the terms of any one of the vertical columns that, if we take, 1ſt, any number of ſucceſſive terms in it, and, 2dly, the ſame number of terms, all equal to the laſt, or greateſt, of the ſaid ſucceſſive terms, the ſum of the ſaid ſucceſſive terms ſhall be to the ſum of the ſame number of terms, all equal to their laſt, or greateſt, term, in the conſtant proportion of 1 to a certain number denoted by the letter r; then it will follow that, if in the next higher vertical column of the ſaid table of figurate numbers we take a number of ſucceſſive terms greater by an unit than the number of ſucceſſive terms taken in the former vertical column, the ſum of theſe ſucceſſive terms in this ſecond column will be to the ſum of the ſame number of terms, all equal to the laſt, or greateſt, of the ſaid ſucceſſive terms, in the proportion of 1 to $r + 1$.

Demonstration.

Let the terms of the former of the two vertical columns be a, b, c, d, e, and f, of which the number is 6, or in general, n; and let the terms of the next higher vertical column be o, g, h, i, l, p, q, of which the number is $n + 1$.

The upper term of this ſecond column is a cypher, o, becauſe every new vertical column of terms muſt have one more cypher preceeding its ſignificant terms than the column immediately preceeding it.

Theſe two vertical columns of terms will be as follows:

	a	o	
	b	g	
	c	h	
n	d	i	$n + 1$,
	e	l	
	f	p	
		q	

If

If theſe columns are the 1ſt and 2d columns of the table, the terms a, b, c, d, e, and f will, each of them, be equal to 1, and g, h, i, l, p, q, will be 1, 2, 3, 4, 5, 6. If theſe columns are the 2d and 3d columns, a will be 0, and b, c, d, e, f will be 1, 2, 3, 4, 5, and g, h, i, l, p, q will be 0, 1, 3, 6, 10, 15. If theſe columns are the 3d and 4th columns, a will be 0, and b will alſo be 0, and c, d, e, f will be 1, 3, 6, 10, and g, h, i, k, p, q will be 0, 0, 1, 4, 10, 20. And in like manner more of the upper terms of both theſe vertical columns will be cyphers, or 0, the farther the columns are taken to the right hand in the table in page 71. But, wherever the columns are taken, the number of terms in them muſt be ſo great as to reach below the cyphers, and take in ſome of the ſignificant terms. Theſe things being premiſed, the demonſtration of this lemma will be as follows.

By the 4th property of the figurate numbers ſet forth above in art. 54, we ſhall have $q = a+b+c+d+e+f$,
and $p = a+b+c+d+e$,
and $l = a+b+c+d$,
and $i = a+b+c$,
and $h = a+b$,
and $g = a$.

And, by the ſuppoſition, the ſum of the n terms a, b, c, d, e, f will be to n times the laſt, or greateſt, term f, as 1 is to r; and the ſum of the $n-1$ terms a, b, c, d, e will be to $n-1$ times the laſt, or greateſt, term e, in the ſame proportion of 1 to r; and the ſum of the $n-2$ terms a, b, c, d will be to $n-2$ times the laſt, or greateſt, term d, in the ſame proportion of 1 to r; and, in like manner, $a+b+c$ will be to $n-3$ times c as 1 to r; and $a+b$ will be to $n-4$ times b as 1 to r; and a will be to $n-5$ times a as 1 to r.

Therefore $a+b+c+d+e+f$ will be $= \frac{nf}{r}$, and $a+b+c+d+e$ will be $= \frac{\overline{n-1}\times e}{r}$, and $a+b+c+d$ will be $= \frac{\overline{n-2}\times d}{r}$, and $a+b+c$ will be $= \frac{\overline{n-3}\times c}{r}$, and

and $a + b$ will be $= \frac{\overline{n-4} \times b}{r}$, and a will be $= \frac{\overline{n-5} \times a}{r}$.

Therefore q (which is equal to $a + b + c + d + e + f$) will be $= \frac{nf}{r}$; and p (which is equal to $a + b + c + d + e$) will be $= \frac{\overline{n-1} \times e}{r}$; and l (which is equal to $a + b + c + d$) will be $= \frac{\overline{n-2} \times d}{r}$; and i (which is equal to $a + b + c$) will be $= \frac{\overline{n-3} \times c}{r}$; and h (which is equal to $a + b$) will be $= \frac{\overline{n-4} \times b}{r}$; and g (which is equal to a) will be $= \frac{\overline{n-5} \times a}{r}$.

Therefore $q+p+l+i+h+g$ will be $= \frac{nf}{r} + \frac{\overline{n-1} \times e}{r} + \frac{\overline{n-2} \times d}{r} + \frac{\overline{n-3} \times c}{r} + \frac{\overline{n-4} \times b}{r} + \frac{\overline{n-5} \times a}{r} = \frac{nf}{r} + \frac{ne - e}{r} + \frac{nd-2d}{r} + \frac{nc-3c}{r} + \frac{nb-4b}{r} + \frac{na-5a}{r} = \frac{nf+ne+nd+nc+nb+na}{r} - \frac{e}{r} - \frac{2d}{r} - \frac{3c}{r} - \frac{4b}{r} - \frac{5a}{r} = \frac{n \times \overline{f+e+d+c+b+a}}{r} \frac{-e-2d-3c-4b-5a}{r} = \frac{n \times q}{r} \frac{-e-2d-3c-4b-5a}{r} = \frac{n \times q}{r} \frac{-e-d-c-b-a}{r} \frac{-d-c-b-a}{r} \frac{-c-b-a}{r} \frac{-b-a}{r} - \frac{a}{r} = \frac{n \times q}{r} - \frac{p}{r} - \frac{l}{r} - \frac{i}{r} - \frac{h}{r} - \frac{g}{r} = \frac{n \times q}{r} \frac{-p-l-i-h-g}{r} = \frac{nq-p-l-i-h-g}{r}$.

Therefore (if we multiply both sides by r) we shall have $r \times \overline{q+p+l+i+h+g} = nq-p-l-i-h-g$, or $rq + r \times \overline{p+l+i+h+g} = nq-p-l-i-h-g$. And consequently (adding $p+l+i+h+g$ to both sides) we shall have $rq + r \times \overline{p+l+i+h+g} + p+l+i+h+g = nq$, or $rq + \overline{r+1} \times \overline{p+l+i+h+g} = nq$; and (subtracting rq from both sides) $\overline{r+1} \times \overline{p+l+i+h+g} = nq - rq$.

Therefore

Therefore (dividing both sides by $r+1$) we shall have $p+l+i+h+g=\frac{nq-rq}{r+1}$; and (adding q to both sides) $q+p+l+i+h+g=\frac{nq-rq}{r+1}+q=\frac{nq-rq}{r+1}+\frac{\overline{r+1}\times q}{r+1}$ $=\frac{nq-rq}{r+1}+\frac{rq+q}{r+1}=\frac{nq+q}{r+1}=\frac{\overline{n+1}\times q}{r+1}$. And consequently $q+p+l+i+h+g$, or $g+h+i+l+p+q$, or $0+g+h+i+l+p+q$, will be to $\overline{n+1}\times q$ as 1 is to $r+1$, or the sum of the $n+1$ successive terms $0, g, h, i, l, p, q$, of the second vertical column of terms will be to the sum of $n+1$ times the last term q, or the same number of terms, all equal to the last, or greatest, term q, in the proportion of 1 to $r+1$ *. Q. E. D.

The principal Proposition, or the 12*th property above-described of the figurate numbers, or numbers contained in the foregoing table of combinations, is as follows.*

76. The sum of any number of terms in any of the vertical columns contained in the foregoing table of combinations is to the sum of the same number of terms all equal to the last term of them, in the proportion of 1 to the exponent of the said column, or to the number which denotes, or expresses, its place in the said table.

Thus, in the first column, of which the exponent is 1, the sum of any number of terms of it denoted by n will be to n times the last term of it in the proportion of 1 to 1, or a proportion of equality. In the second column, of which

* See upon this subject the works of Mr. John Bernoulli himself, published at Lausanne, in Switzerland, in the year 1742, in four volumes, quarto, vol. iii. page 521, in the 47th lecture on the doctrine of the Integration of infinitely small differences, or the Inverse method of differences.

the

the exponent is 2, the ſum of n terms of it will be to the ſum of n terms all equal to the laſt or greateſt term, in the proportion of 1 to 2. In the third column, of which the exponent is 3, the ſum of n terms of it will be to the ſum of n terms all equal to the laſt, or greateſt, term, in the proportion of 1 to 3. In the fourth column, of which the exponent is 4, the ſum of n terms of it will be to the ſum of n terms all equal to the laſt, or greateſt, term, in the proportion of 1 to 4. And, in general, in the cth column, or that of which the exponent is c, the ſum of n terms of it will be to the ſum of n terms all equal to the laſt, or greateſt, or nth, term, in the proportion of 1 to c.

Demonstration.

77. The truth of this propoſition with reſpect to the 1ſt vertical column (which conſiſts wholly of units) is ſhewn above in lemma 1, art. 71, and indeed is almoſt ſelf-evident. And with reſpect to the terms of the ſecond vertical column, to wit, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11, &c, it may be proved by means of the ſecond and fourth of the foregoing lemmas in the manner following. Since in the firſt vertical column the ſum of any number of terms 1, 1, 1, 1, 1, &c. denoted by n, is to the ſum of as many terms, all equal to the laſt term 1, as 1 is to 1; and in the ſecond vertical column the ſum of the two firſt terms 0 and 1 is to the ſum of two terms, both equal to the laſt term 1, as 1 is to $1+1$, as is ſhewn in lemma 2, art. 72; it follows from lemma 4, art. 74, that in the ſame ſecond vertical column the ſum of the three firſt terms 0, 1, and 2, will be to the ſum of three terms all equal to the laſt term 2, in the ſame proportion of 1 to $1+1$, or 2, and conſequently that the ſum of the four firſt terms 0, 1, 2, and 3, will be to the ſum of four terms all equal to the laſt term 3, and the ſum of the five firſt terms 0, 1, 2, 3, and 4, will be to the ſum of five terms all equal to the laſt term 4, and, in general, the ſum of any number of its terms denoted

by n will be to the ſum of n terms all equal to the laſt of them, in the ſame proportion of 1 to 1 + 1, or 2.

Q. E. D.

78. This may likewiſe be proved of the numbers contained in the ſaid ſecond vertical column, independently of the foregoing lemmas, in the manner following.

The numbers contained in the ſaid ſecond vertical column are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c. Now, if we ſet down theſe numbers twice over in two horizontal lines, one under the other, but in contrary orders, ſo that in the ſecond line the laſt term of the firſt line ſhall be placed firſt, and the laſt term but one of the firſt line ſhall be placed ſecond, and ſo on, as in theſe two lines,

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,
11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0,

it is evident that the ſum of every two numbers ſtanding in the ſame vertical line, will be equal to 11, or, in general, to the laſt, or greateſt, term of the ſeries, or (if the ſeries conſiſt of n terms, and conſequently the laſt term be $n - 1$) to $n - 1$. Therefore the ſum of both ſerieſes will be equal to a ſeries conſiſting of the ſame number of terms, or n terms, all equal to the greateſt term $n - 1$; and conſequently the upper ſeries $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 +$ &c . . $n - 1$ alone will be equal to only half of the ſeries of n terms all equal to $n - 1$, or will be to it in the proportion of 1 to 2. Q. E. D.

79. To demonſtrate the ſaid 12th property with reſpect to the numbers contained in the third and fourth and other following vertical columns of the foregoing table of combinations, we muſt have recourſe to the ſecond and fourth lemmas, as in the firſt demonſtration juſt now given of the ſaid property with reſpect to the numbers in the ſecond vertical column. This may be done in the manner following.

By lemma 2d it appears that this property takes place in all the vertical columns, if we continue the terms of each column only till their number is equal to the exponent of the

the column, or ſo as to take in only the firſt ſignificant term of the column, which is always an unit. Therefore in the third vertical column, continued only to the three terms 0, 0, 1, the ſum of the ſaid three terms is to the ſum of three terms all equal to the laſt term 1, in the proportion of 1 to 3, or 1 to 2 + 1. But it has been ſhewn that in the ſecond vertical column 0, 1, 2, 3, 4, 5, 6, 7, &c. it is true univerſally that, whatever be the number we take of its terms, the ſum of the ſaid terms will be to the ſum of as many terms all equal to the laſt term in the proportion of 1 to 2. Here therefore we have the caſe of lemma 4, to wit, that of two contiguous vertical columns, the ſecond and the third, in the former of which the ſum of any number of terms denoted by n is to the ſum of the ſame number of terms all equal to the laſt term in the proportion of 1 to a certain number, which we there denoted by r, and which here is 2, and in the latter of which the ſum of the three firſt terms 0, 0, 1, is to the ſum of three terms all equal to the laſt term 1 as 1 is to $r + 1$, or 2 + 1. It follows therefore from lemma 4th, that, if we take the next term 3 of the ſaid latter vertical column, or continue the ſaid column to four terms, the ſum of the ſaid four terms 0, 0, 1, 3, will alſo be to the ſum of four terms all equal to the laſt term 3 in the ſame proportion of 1 to 2 + 1. And, for the ſame reaſon, the ſum of the five firſt terms of the ſaid third vertical column will be to the ſum of five terms all equal to the fifth term of it, and the ſum of the ſix firſt terms of it will be to the ſum of ſix terms all equal to the ſixth term of it, and, in general, the ſum of any greater number of its terms, denoted by n, will be to the ſum of n terms all equal to the nth term of it, in the ſame proportion of 1 to 2 + 1, or of 1 to 3. Q. E. D.

In like manner we may prove that in the 4th vertical column the ſum of any number n of its terms is to the ſum of n terms all equal to its nth term in the proportion of 1 to 3 + 1, or 1 to 4. For, ſince we have proved that the proportion of theſe two ſums in the third column is that of 1 to 3; and by lemma 2 it appears that, if we take only the

 four

four firſt terms of the 4th column, to wit, 0, 0, 0, 1, the proportion of theſe ſums will be that of 1 to 3 + 1, or 4; it follows from lemma 4th and its corollary, that, if we take five terms of this fourth column, or ſix terms of it, or ſeven terms of it, or, in general, *n* terms of it, the proportion of the ſum of the terms ſo taken to the ſum of the ſame number of terms all equal to the laſt term will always be the ſame proportion of 1 to 3 + 1, or of 1 to 4. Q. E. D.

And by proceeding to apply lemma 2 and lemma 4 in the ſame manner to the fifth, and ſixth, and ſeventh, and other following vertical columns, it may be ſhewn that the proportion of the ſum of any number of terms denoted by *n* to the ſum of *n* terms all equal to the laſt, or greateſt term, will be in the fifth column that of 1 to 4 + 1, or 5, and in the ſixth column that of 1 to 5 + 1, or 6, and, in the ſeventh column that of 1 to 6 + 1, or 7, and, in general, in the *c*th column that of 1 to *c*. Q. E. D.

80. Coroll. 1. In each of the aforeſaid vertical columns of numbers the ſum of any number of the terms beginning with 1, or the firſt ſignificant term of the column, and not reckoning the cyphers that preceed it, as we have hitherto done, will be to the ſum of the ſame number of terms all equal to the next term in the ſaid column after the terms ſo ſummed, in the proportion of 1 to *c*, or the exponent of the column.

Let the terms in the propoſed vertical column, whereof we are to ſum up the ſignificant terms be *a*, *b*, *c*, *d*, &c, *k*, and *l*, including the cyphers, ſo that ſome of the firſt letters *a*, *b*, *c*, &c, ſhall ſtand for cyphers, agreeably to the notation in lemma 4; and let the whole number of theſe terms, including the cyphers, be *n*, agreeably to the ſame notation. And let *m* be the term that comes immediately after *l* the laſt term of the ſet whoſe ſum we are to examine; or, in other words, let *m* be the $\overline{n+1}|$th term of the propoſed vertical column, including the cyphers. Alſo let *c* be the exponent of the ſaid column, and *r* be $= c - 1$. Then

by

by the firſt property of the figurate numbers ſet forth above in art. 51, r will be the number of cyphers in the beginning of the ſaid vertical column, and conſequently $n - r$ will be the number of ſignificant terms in the ſaid column, without the cyphers. We are therefore to prove that the ſum of all the $n - r$ ſignificant terms of the ſaid column a, b, c, d, &c, k and l is to the ſum of $n - r$ terms all equal to the next term m in the proportion of 1 to c or of 1 to $r + 1$.

Now the ſum of all the $n - r$ ſignificant terms of the ſaid vertical column is equal to the ſum of all the n terms of the ſaid column, including the cyphers, becauſe the cyphers are all equal to nothing. And it is ſhewn in the latter part of the demonſtration of lemma 4, that the ſum of the n terms a, b, c, d, &c, k and l is to the next term m in the ſame proportion as $n - r$ is to $r + 1$. Therefore the ſum of all the $n - r$ ſignificant terms of the ſaid column will be to the next term m in the ſame proportion of $n - r$ to $r + 1$. But m is to $\overline{n - r} \times m$ in the ſame proportion as $r + 1$ is to $\overline{n - r} \times \overline{r + 1}$. Therefore, *ex æquo*, the ſum of all the $n - r$ ſignificant terms of the ſaid column will be to $\overline{n - r} \times m$, or to the ſum of $n - r$ terms all equal to the next term m, as $n - r$ is to $\overline{n - r} \times \overline{r + 1}$, and conſequently as 1 is to $r + 1$, or as 1 is to c. Q. E. D.

81. Coroll. 2. By the help of the foregoing corollary we may find the ſum of any given number of terms in any of the vertical columns of the foregoing table of combinations, without actually adding the terms together, by proceeding in the following manner.

Let the number of terms to which the ſeveral vertical columns are continued, be n. Then, as there is one cypher prefixed to the ſignificant terms in the ſecond column, and two cyphers in the third column, and three cyphers in the fourth column, and, in general, $c - 1$ cyphers in the cth column; it is evident that the number of terms in the ſecond column, without the cyphers, will be $n - 1$; and that of of the terms in the third column, without the cyphers, will be $n-2$; and that of the terms in the fourth column, without the cyphers, will be $n-3$; and, in general, that of the terms in the cth column, without the cyphers, will be $n-\overline{c-1}$, or $n-c+1$. The sums of the terms in these several columns may therefore be thus determined.

In the first place, the sum of the n significant terms in the first column, which are all units, will be $n \times 1$, or n.

2dly, The sum of the $n-1$ significant terms in the second column will, by the foregoing corollary, be to $n-1$ times the next following, or $\overline{n+1}|$th, term of the second column as 1 is to 2. But, by the 4th property of the figurate numbers above set forth in art. 54, the $\overline{n+1}|$th term of the second column is equal to the sum of the first n terms of the first column, that is, to n. Therefore the sum of the $n-1$ significant terms of the second column will be to $n-1$ times n, or to $n \times \overline{n-1}$, as 1 is to 2, and consequently will be equal to $\frac{n \times \overline{n-1}}{2}$. Q. E. I.

3dly, The sum of the $n-2$ significant terms in the third column will, by the foregoing corollary, be to $n-2$ times the next following, or $\overline{n+1}|$th, term of the same third column as 1 is to 3. But, by the 4th property of the figurate numbers above set forth, the $\overline{n+1}|$th term of the third column is equal to the sum of the first n terms of the second column, including the cyphers, or (which comes to the same thing) to the sum of the first $n-1$ significant terms of the said second column; which has just now been shewn to be equal to $\frac{n \times \overline{n-1}}{2}$. Therefore the sum of the $n-2$ significant terms in the 3d column will be to $n-2$ times $\frac{n \times \overline{n-1}}{2}$, or to $\frac{n \times \overline{n-1} \times \overline{n-2}}{2}$, as 1 is to 3, and consequently will be equal to $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$. Q. E. I.

4thly,

4thly, The ſum of the $n-3$ ſignificant terms in the fourth column will, by the foregoing corollary, be to $n-3$ times the next following, or $\overline{n+1}$th, term of the ſame fourth column as 1 is to 4. But, by the fourth property of the figurate numbers, the $\overline{n+1}$th term of the fourth column is equal to the ſum of the firſt n terms of the third column, including the cyphers, or of the firſt $n-2$ ſignificant terms of the ſaid third column: which has juſt now been ſhewn to be equal to $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$. Therefore the ſum of the $n-3$ ſignificant terms in the 4th column will be to $n-3$ times $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$, or to $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3}$, as 1 is to 4, and conſequently will be $= \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$. Q. E. I.

5thly, In like manner the ſum of the $n-4$ ſignificant terms in the fifth column will, by the foregoing corollary, be to $n-4$ times the next following, or $\overline{n+1}$th, term of the ſame column as 1 is to 5. But, by the 4th property of the figurate numbers above ſet forth, the $\overline{n+1}$th term of the fifth column is equal to the ſum of the firſt n terms of the fourth column, including the cyphers, or to the firſt $n-3$ ſignificant terms of it; which has juſt now been ſhewn to be $= \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$. Therefore the ſum of the $n-4$ ſignificant terms in the fifth column will be to $n-4$ times $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$, or to $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4}$ as 1 is to 5, and conſequently will be $= \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4 \times 5}$. Q. E. I.

And in like manner it is evident that the ſum of the n firſt terms of the cth column, including the $c-1$ cyphers in the beginning of it, or the ſum of the $n - \overline{c-1}$, or $n-c+1$, first significant terms of the said cth column, will be equal to the fraction $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \&c \times \overline{n-c+1}}{2 \times 3 \times 4 \times 5 \times \&c \times c}$, in the numerator of which the last factor is $n - \overline{c-1}$, or $n - c + 1$, and in the denominator of which the last factor is c.

Q. E. I.

82. Coroll. 3. Since, by the 4th property of the figurate numbers above set forth in art. 54, the $\overline{n+1}$th terms of the second, third, fourth, and other following vertical columns of numbers in the foregoing table of combinations, are respectively equal to the sums of the n first terms of the first, second, third, and other following vertical columns, which sums have been shewn to be, respectively, equal to n, $\frac{n \times \overline{n-1}}{2}$, $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$, &c, it follows that the $\overline{n+1}$th terms of the second, third, fourth, fifth, sixth, and other following vertical columns will be n, $\frac{n \times \overline{n-1}}{2}$, $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$, $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$, and $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4 \times 5}$, &c; and consequently, as every term in the first vertical column is $= 1$, the $\overline{n+1}$th terms of the first, second, third, fourth, fifth, sixth, and other following vertical columns will be 1, n, $\frac{n \times \overline{n-1}}{2}$, $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$, $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$, $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4 \times 5}$, &c, and, in general, the $\overline{n+1}$th term of the cth column will be $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \&c \times \overline{n-c+2}}{2 \times 3 \times 4 \times 5 \; \&c \times \overline{c-1}}$.

83. Coroll. 4. Since the $\overline{n+1}$th terms of the first, second, third, fourth, fifth, sixth, and other following vertical columns of the foregoing table of combinations are 1,

n,

n, $\frac{n\times\overline{n-1}}{2}$, $\frac{n\times\overline{n-1}\times\overline{n-2}}{2\times3}$, $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3\times4}$,
$\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\times3\times4\times5}$, &c,
$\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\ \&c\ \times\overline{n-c+2}}{2\times3\times4\times5\times\ \&c\ \times\overline{c-1}}$, it follows that the nth terms of the ſaid vertical columns will be ſuch as ariſe by ſubſtituting $n-1$ inſtead of n in the foregoing values of the $\overline{n+1}$th terms, and conſequently will be as follows, to wit, 1,
$n-1$, $\frac{\overline{n-1}\times\overline{n-2}}{2}$, $\frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\times3}$, $\frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\times3\times4}$,
$\frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}}{2\times3\times4\times5}$, &c,
$\frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\ \&c\ \overline{n-c+1}}{2\times3\times4\times5\times\ \&c\ \times\overline{c-1}}$.

An Example of the truth of Coroll. 4.

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84. As an example of the truth of this corollary we will derive in this manner the numbers that form the loweſt horizontal row of terms in the foregoing table of combinations, or the twelfth terms of the ſeveral vertical columns of the ſaid table.

Now in this caſe n, or the number of terms in the ſeveral vertical columns, is $=12$. Therefore $n-1$, $n-2$, $n-3$, $n-4$, $n-5$, $n-6$, $n-7$, $n-8$, $n-9$, $n-10$, and $n-11$, are reſpectively equal to 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, and 1. Therefore $n-1$, or the twelfth term of the ſecond vertical column will be $=11$; and $\frac{\overline{n-1}\times\overline{n-2}}{2}$, or the 12th term of the third vertical column, will be $(= \overline{n-1} \times \frac{n-2}{2} = 11 \times \frac{n-2}{2} = 11 \times \frac{10}{2}) = 11 \times 5 = 55$; and $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3}$, or the 12th term of the fourth vertical column, will be $(= \frac{\overline{n-1} \times \overline{n-2}}{2} \times \frac{n-3}{3} = 55 \times \frac{n-3}{3} = 55 \times \frac{9}{3} = 55 \times 3) = 165$; and $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4}$, or the 12th term of the 5th vertical column, will be $(= \frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3} \times \frac{n-4}{4} = 165 \times \frac{n-4}{4} = 165 \times \frac{8}{4} = 165 \times 2) = 330$; and $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5}}{2 \times 3 \times 4 \times 5}$, or the 12th term of the 6th vertical column, will be $(= \frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4} \times \frac{n-5}{5} = 330 \times \frac{n-5}{5} = 330 \times \frac{7}{5} = 66 \times 7) = 462$; and, in like manner, the 12th term of the ſeventh vertical column will be $(= 462 \times \frac{n-6}{6} = 462 \times \frac{6}{6} = 462 \times 1) = 462$; and the 12th term of the eighth vertical column will be $(= 462 \times \frac{n-7}{7} = 462 \times \frac{5}{7} = 66 \times 5) = 330$; and the 12th term of the ninth vertical column will be $(= 330 \times \frac{n-8}{8} = 330 \times \frac{4}{8} = 330 \times \frac{1}{2}) = 165$; and the 12th term of the tenth vertical column will be $(= 165 \times \frac{n-9}{9} = 165 \times \frac{3}{9} = 165 \times \frac{1}{3} = \frac{165}{3}) = 55$; and the 12th term of the eleventh vertical column will be $(= 55 \times \frac{n-10}{10} = 55 \times \frac{2}{10} = 55 \times \frac{1}{5} = \frac{55}{5}) = 11$; and the 12th term of the twelfth, or laſt, vertical column will be $(= 11 \times \frac{n-11}{11} = 11 \times \frac{1}{11}) = 1$. Therefore the 12th terms of the ſaid twelve vertical columns will be as follows, to wit, 1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, and 1; which are the numbers ſet down in the foregoing table.

85. Coroll.

85. Coroll. 5. It has been fhewn above in art. 60 and 61, that the horizontal rows of numbers in the foregoing table of combinations, beginning with the fecond row, exhibit the co-efficients of the terms of the feveral fucceffive powers of a binomial quantity, fuch as $a+b$, every nth horizontal row of numbers being the co-efficients of the terms of the $\overline{n-1}|$th power of the faid binomial quantity; whence it follows that the numbers contained in every $\overline{n+1}|$th horizontal row of numbers in the faid table will be the co-efficients of the terms of the nth power of the faid binomial quantity. But it is evident that the numbers contained in every $\overline{n+1}|$th horizontal row of terms in the faid table are the $\overline{n+1}|$th terms of the firft, fecond, third, fourth, fifth, fixth, and other following vertical columns of terms in the faid table, reckoning the terms from the top of the faid table, and including the cyphers at the tops of all the feveral vertical columns, except the firft. Therefore the $\overline{n+1}|$th terms of the firft, fecond, third, fourth, fifth, fixth, and other following vertical columns of terms in the faid table will be the co-efficients of the terms of the nth power of the faid binomial quantity. But it has been fhewn above in art. 82, coroll. 3, that the $\overline{n+1}|$th terms of the firft, fecond, third, fourth, fifth, and fixth, and other following vertical columns of terms in the faid table are 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c. Therefore the co-efficients of the terms of the nth power of the faid binomial quantity $a+b$ will alfo be 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c; and confequently the quantity $\overline{a+b}|^n$, or the faid nth power itfelf of the faid binomial quantity $a+b$, will be equal to the

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ſeries $a^n + na^{n-1}b + n \times \frac{n-1}{2} a^{n-2}b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3}b^3 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4}b^4 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} a^{n-5}b^5 +$ &c, or (if we put A for 1, and B for n and C for $n \times \frac{n-1}{2}$, and D for $n + \frac{n-1}{2} \times \frac{n-2}{3}$, and E for $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and F for $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, and G, H, I, K, L, &c, for the numeral co-efficients of the ſeventh, eighth, ninth, tenth, eleventh, and other following terms of the ſeries reſpectively) to the ſeries $a^n + \frac{n}{1} A a^{n-1}b + \frac{n-1}{2} B a^{n-2}b^2 + \frac{n-2}{3} C a^{n-3}b^3 + \frac{n-3}{4} D a^{n-4}b^4 + \frac{n-4}{5} E a^{n-5}b^5 + \frac{n-5}{6} F a^{n-6}b^6 + \frac{n-6}{7} G a^{n-7}b^7 + \frac{n-7}{8} H a^{n-8}b^8 + \frac{n-8}{9} I a^{n-9}b^9 + \frac{n-9}{10} K a^{n-10}b^{10} + \frac{n-10}{11} L a^{n-11}b^{11} +$ &c.; which ſeries will continue till the numerator of the generating fraction becomes $n - n$, or 0, and conſequently the ſaid fraction itſelf becomes equal to 0 likewiſe, and therefore the term in which the ſaid fraction enters as a factor, will alſo be equal to 0, as will alſo all the following terms of the ſeries, which would be derived from the ſaid term by continual multiplications. The ſeries therefore will break off, or end with the term next preceeding the ſaid term which is equal to 0 *.

We will now proceed to illuſtrate and confirm the truth of this corollary, by applying the foregoing ſeries to the com-

* This is the famous binomial theorem invented by Sir Iſaac Newton, but of which he has no where given a demonſtration. And the demonſtration here given of it by Mr. James Bernoulli, is that to which I alluded in the firſt volume of the Collection of Tracts, in two volumes, quarto, called *Scriptores Logarithmici*, page 349, art. 4, and in the ſecond volume of the ſame Collection, page 157, art. 9.

computation of the terms of some of the lowest powers of the binomial quantity $a+b$, so as to produce by means of it all the numbers contained in the foregoing table of combinations, in page 74.

Examples of the application of the foregoing series to the computation of the terms of the powers of the binomial quantity a + b.

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86. In the first place let us suppose n to be $= 1$. Then we shall have the series $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \&c$, $(= a^1 + \frac{1}{1} \times 1 \times a^{1-1} b + \frac{1-1}{2} B a^{1-2} b^2 + \frac{1-2}{3} \times C a^{1-3} b^3 + \&c, = a^1 + a^0 \times b + \frac{0}{2} \times B a^{1-2} b^1 + \frac{1-2}{3} \times C \times a^{1-3} b^3 + \&c, = a^1 + 1 \times b + 0 + 0 + \&c,) = a + b$; or the said series is in this case equal to the binomial quantity $a + b$ itself; as it ought to be.

87. Now let n be $= 2$.

Then we shall have the series $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \&c$, $(= a^2 + \frac{2}{1} \times 1 \times a^{2-1} b + \frac{2-1}{2} B a^{2-2} b^2 + \frac{2-2}{3} C a^{2-3} b^3 + \&c, = a^2 + 2a^1 b + \frac{1}{2} \times 2 \times a^0 b^2 + \frac{0}{3} \times C a^{2-3} b^3 + \&c, = a^2 + 2ab + \frac{2}{2} \times 1 \times b^2 + 0 + \&c) = a^2 + 2ab + b^2$; or the said series will in this case be equal to the trinomial quantity

tity $a^2 + 2ab + b^2$; as it ought to be, becaufe that quantity is the fquare of the binomial quantity $a + b$.

88. If n is $= 3$, we fhall have the feries $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \frac{n-3}{4} D a^{n-4} b^4 +$ &c ($= a^3 + \frac{3}{1} \times 1 \times a^{3-1} b + \frac{3-1}{2} \times \frac{3}{1} \times 1 \times a^{3-2} b^2 + \frac{3-2}{3} C a^{3-3} b^3 + \frac{3-3}{4} D a^{3-4} b^4 +$ &c, $= a^3 + 3a^2 b + \frac{2}{2} \times 3 \times a^1 b^2 + \frac{1}{3} C \times a^0 b^3 + \frac{0}{4} D \times a^{3-4} b^4 +$ &c $= a^3 + 3a^2 b + 3ab^2 + \frac{1}{3} \times 3 \times 1 \times b^3 + 0 +$ &c) $= a^3 + 3a^2 b + 3ab^2 + b^3$. Therefore $\overline{a + b}|^n$, or $\overline{a + b}|^3$, or the cube of the binomial quantity $a + b$, will be equal to the quadrinomial quantity $a^3 + 3a^2 b + 3ab^2 + b^3$; as it will be found to be upon trial.

89. If n is $= 4$, we fhall have the feries $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \frac{n-3}{4} D a^{n-4} b^4 + \frac{n-4}{5} E a^{n-5} b^5 +$ &c ($= a^4 + \frac{4}{1} \times 1 \times a^{4-1} b + \frac{4-1}{2} B a^{4-2} b^2 + \frac{4-2}{3} C a^{4-3} b^3 + \frac{4-3}{4} D a^{4-4} b^4 + \frac{4-4}{5} E a^{4-5} b^5 +$ &c $= a^4 + 4a^3 b + \frac{3}{2} \times 4 \times a^2 b^2 + \frac{2}{3} \times C \times a^1 b^3 + \frac{1}{4} D \times a^0 \times b^4 + \frac{0}{5} E \times a^{4-5} b^5 +$ &c $= a^4 + 4a^3 b + 6a^2 b^2 + \frac{2}{3} \times 6 \times ab^3 + \frac{1}{4} D \times 1 \times b^4 + 0 +$ &c. $= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + \frac{1}{4} \times 4 \times b^4 + 0 +$ &c) $= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$. Therefore $\overline{a + b}|^n$, or $\overline{a + b}|^4$, or the fourth power of the binomial quantity $a + b$, will be equal

equal to the quinquinomial quantity $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$; as it will be found to be upon trial.

90. If n is $= 5$, we shall have $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 +$ &c, ($= a^5 + \frac{5}{1} \times 1 \times a^{5-1}b + \frac{5-1}{2} Ba^{5-2}b^2 + \frac{5-2}{3} Ca^{5-3}b^3 + \frac{5-3}{4} Da^{5-4}b^4 + \frac{5-4}{5} Ea^{5-5}b^5 + \frac{5-5}{6} Fa^{5-6}b^6 +$ &c $= a^5 + 5a^4b + \frac{4}{2} Ba^3b^2 + \frac{3}{3} Ca^2b^3 + \frac{2}{4} Da^1b^4 + \frac{1}{5} Ea^0b^5 + \frac{0}{6} \times Fa^{5-6}b^6 +$ &c $= a^5 + 5a^4b + 2Ba^3b^2 + Ca^2b^3 + \frac{1}{2} Da^1b^4 + \frac{1}{5} E \times 1 \times b^5 + 0 +$ &c) $= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$. Therefore $\overline{a+b}|^n$, or $\overline{a+b}|^5$, or the fifth power of the binomial quantity $a+b$, will be equal to the sextinomial quantity $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$; as, upon trial, it will be found to be.

91. If n is $= 6$, we shall have the series $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 +$ &c ($= a^6 + \frac{6}{1} \times 1 \times a^{6-1}b + \frac{6-1}{2} Ba^{6-2}b^2 + \frac{6-2}{3} Ca^{6-3}b^3 + \frac{6-3}{4} Da^{6-4}b^4 + \frac{6-4}{5} Ea^{6-5}b^5 + \frac{6-5}{6} Fa^{6-6}b^6 + \frac{6-6}{7} Ga^{6-7}b^7 +$ &c $= a^6 + 6a^5b + \frac{5}{2} Ba^4b^2 + \frac{4}{3} Ca^3b^3 + \frac{3}{4} Da^2b^4 + \frac{2}{5} Ea^1b^5 + \frac{1}{6} Fa^0b^6 + \frac{0}{7} Ga^{6-7}b^7 +$ &c) $= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. There-

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fore $\overline{a+b}|^n$, or $\overline{a+b}|^6$, or the sixth power of the binomial quantity $a+b$, will be equal to the septinomial quantity $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. And so, upon trial, it will be found to be.

92. If n is $= 7$, we shall have the series $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \frac{n-3}{4} D a^{n-4} b^4 + \frac{n-4}{5} E a^{n-5} b^5 + \frac{n-5}{6} F a^{n-6} b^6 + \frac{n-6}{7} G a^{n-7} b^7 + \frac{n-7}{8} H a^{n-8} b^8 + \&c$ ($= a^7 + \frac{7}{1} \times 1 \times a^6 b + \frac{6}{2} B a^5 b^2 + \frac{5}{3} C a^4 b^3 + \frac{4}{4} D a^3 b^4 + \frac{3}{5} E a^2 b^5 + \frac{2}{6} F a^1 b^6 + \frac{1}{7} G a^0 b^7 + \frac{0}{8} H a^{7-8} b^8 + \&c = a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7ab^6 + 1 \times b^7 + 0 + \&c$) $= a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7ab^6 + b^7$. Therefore $\overline{a+b}|^n$, or $\overline{a+b}|^7$, or the seventh power of the binomial quantity $a+b$, will be equal to the octinomial quantity $a^7 + 7a^6 b + 21a^5 b^2 + 35a^4 b^3 + 35a^3 b^4 + 21a^2 b^5 + 7ab^6 + b^7$. And so, upon trial, it will be found to be.

93. If n is $= 8$, we shall have the series $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \frac{n-3}{4} D a^{n-4} b^4 + \frac{n-4}{5} E a^{n-5} b^5 + \frac{n-5}{6} F a^{n-6} b^6 + \frac{n-6}{7} G a^{n-7} b^7 + \frac{n-7}{8} H a^{n-8} b^8 + \frac{n-8}{9} I a^{n-9} b^9 + \&c$ ($= a^8 + \frac{8}{1} \times 1 \times a^7 b + \frac{7}{2} B a^6 b^2 + \frac{6}{3} C a^5 b^3 + \frac{5}{4} D a^4 b^4 + \frac{4}{5} E a^3 b^5 + \frac{3}{6} F a^2 b^6 + \frac{2}{7} G a^1 b^7 + \frac{1}{8} H a^0 b^8 + \frac{0}{9} I a^{8-9} b^9 + \&c = a^8 + 8a^7 b + 28a^6 b^2 + 56a^5 b^3 + 70a^4 b^4 + 56a^3 b^5 + 28a^2 b^6 + 8ab^7 + 1 \times b^8 + 0 + \&c$) $= a^8 + 8a^7 b + 28a^6 b^2 + 56a^5 b^3 + 70a^4 b^4$

 $56a^3 b^5$

$56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$. Therefore $\overline{a+b}|^n$, or $\overline{a+b}|^8$, or the eighth power of the binomial quantity $a+b$, will be equal to the compound quantity $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$. And so, upon trial, it will be found to be.

94. If n is $= 9$, we shall have the series $a^n + \frac{n}{1}Aa^{n-1}b + \frac{n-1}{2}Ba^{n-2}b^2 + \frac{n-2}{3}Ca^{n-3}b^3 + \frac{n-3}{4}Da^{n-4}b^4 + \frac{n-4}{5}Ea^{n-5}b^5 + \frac{n-5}{6}Fa^{n-6}b^6 + \frac{n-6}{7}Ga^{n-7}b^7 + \frac{n-7}{8}Ha^{n-8}b^8 + \frac{n-8}{9}Ia^{n-9}b^9 + \frac{n-9}{10}Ka^{n-10}b^{10} + \&c$ ($= a^9 + \frac{9}{1} \times 1 \times a^8b + \frac{8}{2}Ba^7b^2 + \frac{7}{3}Ca^6b^3 + \frac{6}{4}Da^5b^4 + \frac{5}{5}Ea^4b^5 + \frac{4}{6}Fa^3b^6 + \frac{3}{7}Ga^2b^7 + \frac{2}{8}Ha^1b^8 + \frac{1}{9}Ia^0b^9 + \frac{0}{10}Ka^{9-10}b^{10} + \&c = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + 1 \times b^9 + 0 + \&c) = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$. Therefore $\overline{a+b}|^n$, or $\overline{a+b}|^9$, or the ninth power of the binomial quantity $a+b$, will be equal to the compound quantity $a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$. And so, upon trial, it will be found to be.

95. If n is $= 10$, we shall have the series $a^n + \frac{n}{1}Aa^{n-1}b + \frac{n-1}{2}Ba^{n-2}b^2 + \frac{n-2}{3}Ca^{n-3}b^3 + \frac{n-3}{4}Da^{n-4}b^4 + \frac{n-4}{5}Ea^{n-5}b^5 + \frac{n-5}{6}Fa^{n-6}b^6 + \frac{n-6}{7}Ga^{n-7}b^7 + \frac{n-7}{8}Ha^{n-8}b^8 + \frac{n-8}{9}Ia^{n-9}b^9 + \frac{n-9}{10}Ka^{n-10}b^{10} + \frac{n-10}{11}$

$La^{n-11}b^{11}$ + &c ($= a^{10} + \frac{10}{1} \times 1 \times a^9b + \frac{9}{2} Ba^8b^2 + \frac{8}{3} Ca^7b^3 + \frac{7}{4} Da^6b^4 + \frac{6}{5} Ea^5b^5 + \frac{5}{6} Fa^4b^6 + \frac{4}{7} Ga^3b^7 + \frac{3}{8} Ha^2b^8 + \frac{2}{9} Ia^1b^9 + \frac{1}{10} Ka^0b^{10} + \frac{0}{11} La^{10-11}b^{11}$ + &c $= a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10a^1b^9 + 1 \times b^{10} + 0 +$ &c) $= a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$. Therefore $\overline{a+b}|^n$, or $\overline{a+b}|^{10}$, or the tenth power of the binomial quantity $a + b$, will be equal to the compound quantity $a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$. And so, upon trial, it will be found to be.

96. Lastly, let n be $= 11$. Then will the series $a^n + \frac{n}{1} Aa^{n-1}b + \frac{n-1}{2} Ba^{n-2}b^2 + \frac{n-2}{3} Ca^{n-3}b^3 + \frac{n-3}{4} Da^{n-4}b^4 + \frac{n-4}{5} Ea^{n-5}b^5 + \frac{n-5}{6} Fa^{n-6}b^6 + \frac{n-6}{7} Ga^{n-7}b^7 + \frac{n-7}{8} Ha^{n-8}b^8 + \frac{n-8}{9} Ia^{n-9}b^9 + \frac{n-9}{10} Ka^{n-10}b^{10} + \frac{n-10}{11} La^{n-11}b^{11} + \frac{n-11}{12} Ma^{n-12}b^{12}$ + &c be ($= a^{11} + \frac{11}{1} \times 1 \times a^{10}b + \frac{10}{2} Ba^9b^2 + \frac{9}{3} Ca^8b^3 + \frac{8}{4} Da^7b^4 + \frac{7}{5} Ea^6b^5 + \frac{6}{6} Fa^5b^6 + \frac{5}{7} Ga^4b^7 + \frac{4}{8} Ha^3b^8 + \frac{3}{9} Ia^2b^9 + \frac{2}{10} Ka^1b^{10} + \frac{1}{11} La^0b^{11} + \frac{0}{12} Ma^{11-12}b^{12}$ + &c $= a^{11} + 11a^{10}b + 55a^9b^2 + 165a^8b^3 + 330a^7b^4 + 462a^6b^5 + 462a^5b^6 + 330a^4b^7 + 165a^3b^8 + 55a^2b^9 + 11ab^{10} + 1 \times b^{11} + 0 +$ &c) $= a^{11} + 11a^{10}b + 55a^9b^2 + 165a^8b^3 + 330a^7b^4 + 462a^6b^5 + 462a^5b^6 + 330a^4b^7 + 165a^3b^8 + 55a^2b^9 + 11ab^{10} + b^{11}$. Therefore

fore $\overline{a+b}|^n$, or $\overline{a+b}|^{11}$, or the eleventh power of the binomial quantity $a+b$, will be equal to the compound quantity $a^{11} + 11a^{10}b + 55a^9b^2 + 165a^8b^3 + 330a^7b^4 + 462a^6b^5$ $462a^5b^6 + 330a^4b^7 + 165a^3b^8 + 55a^2b^9 + 11ab^{10} + b^{11}$. And so, upon trial, it will be found to be.

97. It appears therefore that the series $a^n + \frac{n}{1} A a^{n-1} b + \frac{n-1}{2} B a^{n-2} b^2 + \frac{n-2}{3} C a^{n-3} b^3 + \frac{n-3}{4} D a^{n-4} b^4 + \frac{n-4}{5} E a^{n-5} b^5 + \frac{n-5}{6} F a^{n-6} b^6 +$ &c, which has been obtained in the foregoing 5th corollary for the value of the quantity $\overline{a+b}|^n$, or the nth power of the binomial quantity $a+b$, does truly exhibit the value of the said power when the index n is equal to either 1, or 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, or 10, or 11; in which cases the co-efficients of the terms of the said series are equal to the numbers contained in the several successive horizontal rows of terms in the foregoing table of combinations, in page 74, beginning with the second row.

Additional Corollaries, not contained in the original text of Mr. James Bernoulli.

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98. To these five corollaries, which are contained in Mr. James Bernoulli's original text, it may not be amiss to add the following corollaries, which are easily deducible from Mr. Bernoulli's propositions, and which will enable us to find a general expression for the terms of any of the verti-

cal columns in the foregoing table of combinations, or, in other words, for the figurate numbers of any propoſed order.

99. Coroll. 6. It has been ſhewn in art. 82, corol. 3, page 112, that the $\overline{n+1}|$th terms of the firſt, ſecond, third, fourth, fifth, ſixth, and other following vertical columns of terms in the foregoing table of combinations are 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c. But the $\overline{n+1}|$th terms of the firſt, ſecond, third, fourth, fifth, ſixth, and other following vertical columns of terms in the ſaid table are the firſt, ſecond, third, fourth, fifth, ſixth, and other following terms of the $\overline{n+1}|$th horizontal row of terms in the ſaid table. Therefore the firſt, ſecond, third, fourth, fifth, ſixth, and other following terms of the $\overline{n+1}|$th horizontal row of terms in the ſaid table are 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c.

100. Coroll. 7. Since the ſeveral terms of the $\overline{n+1}|$th horizontal row are 1, n, $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c, it follows that, if p be any whole number greater than n, as, for example, $n+1$, or $n+2$, or $n+3$, or $n+4$, &c, the ſeveral terms of the $\overline{p+1}|$th horizontal row will be 1, p, $p \times \frac{p-1}{2}$, $p \times \frac{p-1}{2} \times \frac{p-2}{3}$, $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}$, and $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}$, &c. And conſequently the third term of the $\overline{p+1}|$th horizontal row of terms, when p is equal to $n+1$, or the third term of the $\overline{n+2}|$th horizontal row

row of terms, will be the quantity which arises by substituting $n+1$ instead of p in the third term, $p \times \frac{p-1}{2}$, of the last-mentioned series, that is, $\overline{n+1} \times \frac{n+1-1}{2}$, or $\overline{n+1} \times \frac{n}{2}$, or $n \times \frac{n+1}{2}$; and the fourth term of the $\overline{p+1}|$th horizontal row of terms when p is $= n+2$, or the fourth term of the $\overline{n+3}|$th horizontal row of terms will be that which arises by substituting $n+2$ instead of p in the fourth term, $p \times \frac{p-1}{2} \times \frac{p-2}{3}$, of the last series, that is, $\overline{n+2} \times \frac{n+2-1}{2} \times \frac{n+2-2}{3}$, or $\overline{n+2} \times \frac{n+1}{2} \times \frac{n}{3}$, or $n \times \frac{n+1}{2} \times \frac{n+2}{3}$.

And, in like manner, if we substitute $n+3$ instead of p in the 5th term $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}$ of the last series, we shall have $\overline{n+3} \times \frac{n+3-1}{2} \times \frac{n+3-2}{3} \times \frac{n+3-3}{4}$, or $\overline{n+3} \times \frac{n+2}{2} \times \frac{n+1}{3} \times \frac{n}{4}$, or $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, for the 5th term of the $\overline{n+4}|$th horizontal row of terms; and, if we substitute $n+4$ instead of p in the 6th term, $p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}$, of the last series, we shall have $\overline{n+4} \times \frac{n+4-1}{2} \times \frac{n+4-2}{3} \times \frac{n+4-3}{4} \times \frac{n+4-4}{5}$, or $\overline{n+4} \times \frac{n+3}{2} \times \frac{n+2}{3} \times \frac{n+1}{4} \times \frac{n}{5}$, or $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$, for the 6th term of the $\overline{n+5}|$th horizontal row of terms. So that the 3d term of the $\overline{n+2}|$th horizontal row of terms, and the 4th term of the $\overline{n+3}|$th horizontal row, and the 5th term of the $\overline{n+4}|$th horizontal row, and the 6th term of the $\overline{n+5}|$th horizontal row, will be $n \times \frac{n+1}{2}$, $n \times$

$\frac{n+1}{2}$

$\frac{n+1}{2} \times \frac{n+2}{3}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, and $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$, respectively.

101. Coroll. 8. These four terms $n \times \frac{n+1}{2}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3}$, $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, and $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$, are derived from the number n by the continual multiplication of the fractions $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, and $\frac{n+4}{5}$, the numerators and denominators of which both increase continually by an unit. Therefore, if we put C for the first, D for the second, E for the third, and F for the fourth of these terms, we shall have $C = \frac{n+1}{2} \times n$, and $D = \frac{n+2}{3} \times C$, and $E = \frac{n+3}{4} \times D$, and $F = \frac{n+4}{5} \times E$.

And, from the manner in which these four terms were derived from the 3d, 4th, 5th, and 6th terms of the series $1, p, p \times \frac{p-1}{2}, p \times \frac{p-1}{2} \times \frac{p-2}{3}, p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}, p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}$, &c, in the last corollary, to wit, by substituting $n+1$, $n+2$, $n+3$, and $n+4$, instead of p in the said 3d, 4th, 5th, and 6th terms, respectively, it is easy to see that the 7th term of the $\overline{n+6}|$th horizontal row of terms, and the 8th term of the $\overline{n+7}|$th horizontal row, and the 9th term of the $\overline{n+8}|$th horizontal row, and the 10th term of the $\overline{n+9}|$th horizontal row, and the 11th, 12th, 13th, 14th, 15th, and other following terms of the $\overline{n+10}|$th, $\overline{n+11}|$th, $\overline{n+12}|$th, $\overline{n+13}|$th, $\overline{n+14}|$th, and other following horizontal rows of terms in the said table, respectively, will be equal to $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$,

$\times$ H, $\frac{n+8}{9} \times$ I, $\frac{n+9}{10} \times$ K, $\frac{n+10}{11} \times$ L, $\frac{n+11}{12} \times$ M, $\frac{n+12}{13} \times$ N, $\frac{n+13}{14} \times$ O, &c, in which the capital letters G, H, I, K, L, M, N, O, &c, denote the 7th, 8th, 9th, 10th, 11th, 12th, 13th, 14th, 15th, and other next following terms, of the $\overline{n+6}|$th, $\overline{n+7}|$th, $\overline{n+8}|$th, $\overline{n+9}|$th, $\overline{n+10}|$th, and other following horizontal rows of terms, reſpectively, as they ariſe, and the generating fractions $\frac{n+5}{6}$, $\frac{n+6}{7}$, $\frac{n+7}{8}$, $\frac{n+8}{9}$, $\frac{n+9}{10}$, $\frac{n+10}{11}$, $\frac{n+11}{12}$, $\frac{n+12}{13}$, $\frac{n+13}{14}$, &c, are a continuation of the foregoing generating fractions $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, and are derived from them by the continual addition of an unit to both their numerators and denominators.

102. Coroll. 9. It is ſhewn above in the 6th property of the numbers contained in the foregoing table of combinations, art. 56 and 57, pages 77 and 78, that in every horizontal row of terms in the ſaid table of combinations, the firſt and laſt term are, each of them, an unit, and the terms that are equidiſtant from the firſt and laſt terms are equal to each other. It follows therefore that the 3d term of the $\overline{n+2}|$th horizontal row, reckoned from the end of it, or from the right hand to the left, will be equal to the 3d term of it reckoned from the beginning, or from the left hand to the right; and that the 4th term of the $\overline{n+3}|$th horizontal row, reckoned from the end of it, or from the right hand to the left, will be equal to the 4th term of it reckoned from the beginning, or from the left hand to the right; and that the 5th term of the $\overline{n+4}|$th horizontal row, reckoned from the end of it, or from the right hand to the left, will be equal to the 5th term of it, reckoned from the beginning, or from the left hand to the right; and that the 6th term of the $\overline{n+5}|$th horizontal row, reckoned from the end of it, or from the right hand to the left, will be equal to the 6th term of it, reckoned from the beginning, or from the left hand to the right; and, in like manner, that the

 7th,

7th, and 8th, and 9th, and 10th, and other following terms of the $\overline{n+6}|$th, $\overline{n+7}|$th, $\overline{n+8}|$th, $\overline{n+9}|$th, and other following horizontal rows of terms, respectively, reckoned from the ends of the said rows, or from the right hand to the left, will be equal to the 7th, and 8th, and 9th, and 10th, and other following corresponding terms of the same horizontal rows, respectively, reckoned from the beginnings of the said rows, or from the left hand to the right. But it was shewn in corollary 7th, that the 3d term of the $\overline{n+2}|$th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2}$; and that the 4th term of the $\overline{n+3}|$th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2} \times \frac{n+2}{3}$; and that the 5th term of the $\overline{n+4}|$th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$; and that the sixth term of the $\overline{n+5}|$th horizontal row of terms, reckoned from the beginning of it, or from the left hand to the right, is $n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}$: or that, if the said third term of the $\overline{n+2}|$th horizontal row of terms, reckoned from the beginning of it, be called C, and the said 4th term of the $\overline{n+3}|$th horizontal row of terms, reckoned from the beginning of it, be called D, and the said 5th term of the $\overline{n+4}|$th horizontal row of terms, reckoned from the beginning of it, be called E, and the said 6th term of the $\overline{n+5}|$th horizontal row of terms, reckoned from the beginning of it, be called F, we shall have $C = n \times \frac{n+1}{2}$, and $D = \frac{n+2}{3} \times C$, and $E = \frac{n+3}{4} \times D$, and $F = \frac{n+4}{5} \times E$. And it is observed in the last, or 8th, corollary, that, if the 7th term of the $\overline{n+6}|$th horizontal

rizontal row of terms be called G, and the 8th term of the $\overline{n+7}|$th horizontal row be called H, and the 9th term of the $\overline{n+8}|$th horizontal row, and the 10th term of the $\overline{n+9}|$th horizontal row, and the 11th term of the $\overline{n+10}|$th horizontal row, and the 12th term of the $\overline{n+11}|$th horizontal row, and the next following terms of the next following horizontal rows, all reckoned from the beginnings of thoſe ſeveral horizontal rows, or from the left hand to the right, be called I, K, L, and M, &c, reſpectively, we ſhall have G $= \frac{n+5}{6} \times$ F, and H $= \frac{n+6}{7} \times$ G, and I $= \frac{n+7}{8} \times$ H, and K $= \frac{n+8}{9} \times$ I, and L $= \frac{n+9}{10} \times$ K, and M $= \frac{n+10}{11} \times$ L, and conſequently that the 3d term of the $\overline{n+2}|$th horizontal row of terms, and the 4th term of the $\overline{n+3}|$th horizontal row of terms, and the 5th, 6th, 7th, 8th, 9th, 10th, 11th, and 12th, and other next following terms of the $\overline{n+4}|$th, $\overline{n+5}|$th, $\overline{n+6}|$th, $\overline{n+7}|$th, $\overline{n+8}|$th, $\overline{n+9}|$th, $\overline{n+10}|$th, and $\overline{n+11}|$th, and other next following horizontal rows of terms, reckoned from the beginnings of thoſe ſeveral horizontal rows, or from the left hand to the right, will be equal to $n \times \frac{n+1}{2}$, or C, and $\frac{n+2}{3} \times$ C, $\frac{n+3}{4} \times$ D, $\frac{n+4}{5} \times$ E, $\frac{n+5}{6} \times$ F, $\frac{n+6}{7} \times$ G, $\frac{n+7}{8} \times$ H, $\frac{n+8}{9} \times$ I, $\frac{n+9}{10} \times$ K, and $\frac{n+10}{11} \times$ L, &c, reſpectively. It follows therefore that the 3d term of the $\overline{n+2}|$th horizontal row of terms, and the 4th term of the $\overline{n+3}|$th horizontal row of terms, and the 5th, 6th, 7th, 8th, 9th, 10th, 11th, and 12th, and other next following terms of the $\overline{n+4}|$th, $\overline{n+5}|$th, $\overline{n+6}|$th, $\overline{n+7}|$th, $\overline{n+8}|$th, $\overline{n+9}|$th, $\overline{n+10}|$th, and $\overline{n+11}|$th, and other next following horizontal rows of terms, reckoned from the ends of thoſe ſeveral horizontal rows, or from the right

right hand to the left, will also be respectively equal to $n \times \frac{n+1}{2}$, or C, and $\frac{n+2}{3} \times C$, $\frac{n+3}{4} \times D$, $\frac{n\times 4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$, $\frac{n+8}{9} \times I$, $\frac{n+9}{10} \times K$, and $\frac{n+10}{11} \times L$, &c.

Of the figurate numbers, or the significant terms of the vertical columns of terms in the foregoing table of combinations, page 74.

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103. Coroll. 10. We come now to consider the vertical columns of terms in the foregoing table of combinations.

Now it is evident, in the first place, that the first significant term in every vertical column of terms in the said table is an unit, and that the second significant term is the number which is the exponent of the column; as has been observed above in art. 53, page 76. So that, if the whole number n be the exponent of the column, the two first significant terms of the said column, immediately following the cyphers at the top of it, will be 1 and n. It remains that we find the values of the following terms in the said column, after the terms 1 and n. Now this may be done by means of the foregoing corollaries, in the manner following.

In the foregoing table of combinations the number of cyphers at the top of the vertical column of which the exponent is n, is $n-1$; as is observed above in art. 51, page 75. And consequently the first significant term in the said vertical column, to wit, 1, will be the nth term of it, and consequently will be situated in the nth horizontal row of terms in the said table; and the second significant term in the said vertical column of terms, to wit, n, will be situated

in

in the $\overline{n+1}$th horizontal row of terms; and the 3d ſignificant term in the ſaid vertical column will be ſituated in the $\overline{n+2}$th horizontal row of terms; and the 4th ſignificant figure in the ſaid vertical column will be ſituated in the $\overline{n+3}$th horizontal row of terms; and, in like manner, the 5th, and 6th, and 7th, and 8th, and 9th, and 10th, and other following ſignificant terms in the ſaid nth vertical column will be ſituated in the $\overline{n+4}$th, and $\overline{n+5}$th, and $\overline{n+6}$th, and $n+7$th, and $\overline{n+8}$th, and $\overline{n+9}$th, and other following horizontal rows of terms reſpectively.

And, further, the firſt ſignificant term, to wit, 1, in the ſaid nth vertical column of terms, is likewiſe the firſt term, reckoned from the right hand to the left, of the horizontal row in which it is ſituated; and the ſecond ſignificant term in the ſaid nth vertical column of terms, to wit, n, is likewiſe the ſecond term, reckoned from the right hand to the left, of the horizontal row in which it is ſituated; and the 3d ſignificant term in the ſaid nth vertical column is likewiſe the third term, reckoned from the right hand to the left, of the horizontal row in which it is ſituated; and the 4th ſignificant term of the ſaid nth vertical column is likewiſe the 4th term, reckoned from the right hand to the left, of the horizontal row in which it is ſituated; and, in like manner, the 5th, 6th, 7th, 8th, and other following ſignificant terms of the ſaid nth vertical column of terms, are likewiſe the 5th, 6th, 7th, 8th, and other following terms, reckoned from the right hand to the left, of the ſeveral horizontal rows of terms in which they are ſituated, reſpectively.

But it has been ſhewn that the 1ſt, 2d, 3d, 4th, 5th, 6th, and other following ſignificant terms in the nth vertical column of terms are ſituated in the nth, $\overline{n+1}$th, $\overline{n+2}$th, $\overline{n+3}$th, $\overline{n+4}$th, $\overline{n+5}$th, and other next following horizontal rows of terms, reſpectively.

Therefore the 1ſt, 2d, 3d, 4th, 5th, 6th, and other following ſignificant terms of the nth vertical column of terms are likewise the 1st, 2d, 3d, 4th, 5th, 6th, and other next following terms, reckoned from the right hand to the left, of the nth, $\overline{n+1}$th, $\overline{n+2}$th, $\overline{n+3}$th, $\overline{n+4}$th, $\overline{n+5}$th, and other next following horizontal rows of terms, respectively.

But it has been shewn in coroll. 9, that the 3d term of the $\overline{n+2}$th horizontal row of terms, reckoned from the right hand to the left, is equal to $n \times \frac{n+1}{2}$, or $\frac{n+1}{2} \times n$, or $\frac{n+1}{2} \times B$; and that the 4th term of the $\overline{n+3}$th horizontal row of terms, reckoned from the right hand to the left, is $n \times \frac{n+1}{2} \times \frac{n+2}{3}$, or $\frac{n+2}{3} \times C$; and that the 5th term of the $\overline{n+4}$th horizontal row of terms, and the 6th term of the $\overline{n+5}$th horizontal row of terms, and the 7th, 8th, 9th, 10th, and other next following terms of the $\overline{n+6}$th, $\overline{n+7}$th, $\overline{n+8}$th, $\overline{n+9}$th, and other next following horizontal rows of terms, respectively, all reckoned from the right hand to the left, are equal to $\frac{n+3}{4} \times D$, $\frac{n+4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, &c.

Therefore the 3d significant term of the nth vertical column of terms will be equal to $n \times \frac{n+1}{2}$, or $\frac{n+1}{2} \times n$, or $\frac{n+1}{2} \times B$; and the 4th significant term of the same vertical column will be equal to $n \times \frac{n+1}{2} \times \frac{n+2}{3}$, or $\frac{n+2}{3} \times C$; and the 5th, and 6th, and 7th, and 8th, and 9th, and 10th, and other following significant terms of the same vertical column will be equal to $\frac{n+3}{4} \times D$, $\frac{n+4}{5} \times E$, $\frac{n+5}{6} \times F$, $\frac{n+6}{7} \times G$, $\frac{n+7}{8} \times H$, $\frac{n+8}{9} \times I$, &c.; and consequently the

whole

whole of the said nth vertical column of terms, including the two first significant terms 1 and n, or, in other words, the whole series of figurate numbers of the nth order, will be 1, n, or $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, $\frac{n+8}{9}$ I, $\frac{n+9}{10}$ K, $\frac{n+10}{11}$ L, $\frac{n+11}{12}$ M, $\frac{n+12}{13}$ N, $\frac{n+13}{14}$ O, $\frac{n+14}{15}$ P, &c, *ad infinitum.* Q. E. I.

Examples of the application of the foregoing series to the computation of the figurate numbers of several successive orders.

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104. In the first place we will suppose the letter n to denote 1.

Then we shall have $n+1$ $(= 1+1) = 2$, and $n+2$ $(= 1+2) = 3$, and $n+3$ $(= 1+3) = 4$, and $n+4$ $(= 1+4) = 5$, and $n+5$, $n+6$, $n+7$, $n+8$, $n+9$, $n+10$, &c, $=$ 6, 7, 8, 9, 10, 11, &c, respectively. Therefore the several terms 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, $\frac{n+8}{9}$ I, $\frac{n+9}{10}$ K, $\frac{n+10}{11}$ L, &c, will, in this case, be equal to 1, $\frac{1}{1}$ A, $\frac{2}{2}$ B, $\frac{3}{3}$ C, $\frac{4}{4}$ D, $\frac{5}{5}$ E, $\frac{6}{6}$ F, $\frac{7}{7}$ G, $\frac{8}{8}$ H, $\frac{9}{9}$ I, $\frac{10}{10}$ K, $\frac{11}{11}$ L, &c, respectively, or to 1, 1 A, 1 B, 1 C, 1 D, 1 E, 1 F, 1 G, 1 H, 1 I, 1 K, 1 L, &c, or to 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c; and therefore the twelve first terms in the first vertical column in the foregoing table of combinations, or the twelve first figurate numbers of the 1st order, obtained by means

means of the foregoing ſeries, will be 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, and 1, or a ſet of units; as they ought to be, and as they are in the foregoing table of combinations.

105. In the next place we will ſuppoſe the exponent n to be equal to 2, in order to obtain, by means of the foregoing ſeries, the ſignificant terms in the ſecond vertical column in the foregoing table of combinations, or the figurate numbers of the ſecond order.

Now, if n is $= 2$, the terms of the ſeries $1, \frac{n}{1}A, \frac{n+1}{2}B, \frac{n+2}{3}C, \frac{n+3}{4}D, \frac{n+4}{5}E, \frac{n+5}{6}F, \frac{n+6}{7}G, \frac{n+7}{8}H, \frac{n+8}{9}I, \frac{n+9}{10}K$, &c, will be reſpectively equal to $1, \frac{2}{1}A, \frac{2+1}{2}B, \frac{2+2}{3}C, \frac{2+3}{4}D, \frac{2+4}{5}E, \frac{2+5}{6}F, \frac{2+6}{7}G, \frac{2+7}{8}H, \frac{2+8}{9}I, \frac{2+9}{10}K$, &c, or $1, \frac{2}{1}A, \frac{3}{2}B, \frac{4}{3}C, \frac{5}{4}D, \frac{6}{5}E, \frac{7}{6}F, \frac{8}{7}G, \frac{9}{8}H, \frac{10}{9}I, \frac{11}{10}K$, &c, or 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c. Therefore the firſt eleven ſignificant terms in the ſecond vertical column of terms in the foregoing table of combinations, or the firſt eleven figurate numbers of the ſecond order, obtained by means of the foregoing ſeries, will be the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11; as they ought to be.

106. In the 3d place we will ſuppoſe the exponent n to be equal to 3, in order to obtain, by means of the foregoing ſeries, the ſignificant terms in the 3d vertical column of terms in the foregoing table of combinations, or the figurate numbers of the third order, or (as they are often called) the *triangular* numbers.

Now, if n is $= 3$, the terms of the ſeries $1, \frac{n}{1}A, \frac{n+1}{2}B, \frac{n+2}{3}C, \frac{n+3}{4}D, \frac{n+4}{5}E, \frac{n+5}{6}F, \frac{n+6}{7}G, \frac{n+7}{8}H, \frac{n+8}{9}I$, &c,

&c, will be respectively equal to 1, $\frac{3}{1}$ A, $\frac{3+1}{2}$ B, $\frac{3+2}{3}$ C, $\frac{3+3}{4}$ D, $\frac{3+4}{5}$ E, $\frac{3+5}{6}$ F, $\frac{3+6}{7}$ G, $\frac{3+7}{8}$ H, $\frac{3+8}{9}$ I, &c, or 1, $\frac{3}{1}$ A, $\frac{4}{2}$ B, $\frac{5}{3}$ C, $\frac{6}{4}$ D, $\frac{7}{5}$ E, $\frac{8}{6}$ F, $\frac{9}{7}$ G, $\frac{10}{8}$ H, $\frac{11}{9}$ I, &c, or 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, &c. Therefore the first ten significant terms in the third vertical column of terms in the foregoing table of combinations, or the first ten figurate numbers of the third order, or the first ten triangular numbers, obtained by means of the foregoing series, are 1, 3, 6, 10, 15, 21, 28, 36, 45, and 55; which are the same numbers with those set down above in page 74, in the third vertical column of the foregoing table of combinations.

107. In the 4th place we will suppose the exponent n to be $= 4$, in order to obtain, by means of the foregoing series, the significant terms in the 4th vertical column in the foregoing table of combinations, or the figurate numbers of the 4th order, or (as they are often called) the *pyramidal* numbers.

Now, if n is $= 4$, the terms of the series 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, &c, will be respectively equal to 1, $\frac{4}{1}$ A, $\frac{4+1}{2}$ B, $\frac{4+2}{3}$ C, $\frac{4+3}{4}$ D, $\frac{4+4}{5}$ E, $\frac{4+5}{6}$ F, $\frac{4+6}{7}$ G, $\frac{4+7}{8}$ H, &c, or 1, $\frac{4}{1}$ A, $\frac{5}{2}$ B, $\frac{6}{3}$ C, $\frac{7}{4}$ D, $\frac{8}{5}$ E, $\frac{9}{6}$ F, $\frac{10}{7}$ G, $\frac{11}{8}$ H, &c, or 1, 4, 10, 20, 35, 56, 84, 120, 165, &c. Therefore the first nine significant terms in the fourth vertical column of terms in the foregoing table of combinations, or the first nine figurate numbers of the 4th order, or the first nine pyramidal numbers, obtained by means of the foregoing series, are 1, 4, 10, 20, 35, 56, 84, 120, and 165; which are the same numbers

bers with those set down above in page 74, in the fourth vertical column of the foregoing table of combinations.

108. In the 5th place we will suppose the exponent n to be $= 5$, in order to obtain, by means of the foregoing series, the significant terms in the 5th vertical column of terms in the foregoing table of combinations, or the figurate number of the 5th order.

Now, if n is $= 5$, the terms of the series $1, \frac{n}{1}A, \frac{n+1}{2}B, \frac{n+2}{3}C, \frac{n+3}{4}D, \frac{n+4}{5}E, \frac{n+5}{6}F, \frac{n+6}{7}G$, &c, will be respectively equal to $1, \frac{5}{1}A, \frac{5+1}{2}B, \frac{5+2}{3}C, \frac{5+3}{4}D, \frac{5+4}{5}E, \frac{5+5}{6}F, \frac{5+6}{7}G$, &c, or $1, \frac{5}{1}A, \frac{6}{2}B, \frac{7}{3}C, \frac{8}{4}D, \frac{9}{5}E, \frac{10}{6}F, \frac{11}{7}G$, &c, or 1, 5, 15, 35, 70, 126, 210, 330, &c. Therefore the first eight significant terms in the 5th vertical column of terms in the foregoing table of combinations, or the first eight figurate numbers of the 5th order, are 1, 5, 15, 35, 70, 126, 210, and 330; which are the same numbers with those set down above in page 74 in the 5th vertical column of the foregoing table of combinations.

109. In like manner, if the exponent n is $= 6$, the terms of the series $1, \frac{n}{1}A, \frac{n+1}{2}B, \frac{n+2}{3}C, \frac{n+3}{4}D, \frac{n+4}{5}E, \frac{n+5}{6}F$, &c, will be respectively equal to $1, \frac{6}{1}A, \frac{7}{2}B, \frac{8}{3}C, \frac{9}{4}D, \frac{10}{5}E, \frac{11}{6}F$, &c, or 1, 6, 21, 56, 126, 252, 462, &c. and consequently the first seven significant terms of the 6th vertical column of terms in the foregoing table, or the first seven figurate numbers of the 6th order, will be 1, 6, 21, 56, 126, 252, and 462; which are the same numbers with those set down above in page 74 in the 6th vertical column of the foregoing table of combinations.

110. And,

110. And, if the exponent n is $= 7$, the terms of the ſeries $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D, \frac{n+4}{5} E$, &c, will be reſpectively equal to $1, \frac{7}{1} A, \frac{8}{2} B, \frac{9}{3} C, \frac{10}{4} D, \frac{11}{5} E$, &c, or 1, 7, 28, 84, 210, 462, &c; and conſequently the firſt ſix ſignificant terms of the 7th vertical column of terms in the foregoing table of combinations, or the firſt ſix figurate numbers of the 7th order, will be 1, 7, 28, 84, 210, and 462; which are the ſame numbers with thoſe ſet down above in page 74, in the 7th vertical column of the foregoing table of combinations.

111. And, if the exponent n is $= 8$, the terms of the ſeries $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C, \frac{n+3}{4} D$, &c, will be reſpectively equal to $1, \frac{8}{1} A, \frac{9}{2} B, \frac{10}{3} C, \frac{11}{4} D$, &c, or 1, 8, 36, 120, 330, &c; and conſequently the firſt five ſignificant terms of the 8th vertical column of terms in the foregoing table of combinations, or the firſt five figurate numbers of the 8th order, will be 1, 8, 36, 120, and 330; which are the ſame numbers with thoſe ſet down above in page 74, in the 8th vertical column of the foregoing table of combinations.

112. And, if the exponent n is $= 9$, the terms of the ſeries $1, \frac{n}{1} A, \frac{n+1}{2} B, \frac{n+2}{3} C$, &c, will be reſpectively equal to $1, \frac{9}{1} A, \frac{10}{2} B, \frac{11}{3} C$, &c, or 1, 9, 45, 165, &c; and conſequently the firſt four ſignificant terms in the 9th vertical column of terms in the foregoing table of combinations, or the firſt four figurate numbers of the 9th order, will be 1, 9, 45, and 165; which are the ſame numbers with thoſe ſet down above in page 74, in the 9th vertical column of the foregoing table of combinations.

113. And, if the exponent n is $=$ 10, the terms 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, &c, will be respectively equal to 1, $\frac{10}{1}$ A, $\frac{11}{2}$ B, &c, or 1, 10, 55, &c; and consequently the three first significant terms of the 10th vertical column of terms in the foregoing table of combinations, or the three first figurate numbers of the 10th order, will be 1, 10, and 55; which are the same numbers with those set down above in page 74, in the 10th vertical column of the said table of combinations.

114. It appears, therefore, that all the numbers set down above in page 74, in the ten first vertical columns of the foregoing table of combinations, may be obtained by the application of the general series 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, $\frac{n+8}{9}$ I, $\frac{n+9}{10}$ K, $\frac{n+10}{11}$ L, &c; which is a confirmation of the truth of the said series, and of the reasonings by which it was obtained.

A general

A general expreſſion of the value of the fraction $\frac{1}{\overline{a+b}|^n}$, *or the reciprocal of any integral power of the binomial quantity* a + b, *in an infinite ſeries.*

---

115. Coroll. 11. From the foregoing corollary we may derive a general expreſſion for the value of the quantity $\overline{a+b}|^{-n}$, or $\frac{1}{\overline{a+b}|^n}$, in an infinite ſeries of terms, when the index n is any whole number whatſoever.

For the quantity $\frac{1}{\overline{a+b}|^n}$ is equal to the ſeries which reſults from the diviſion of the numerator 1 by the binomial quantity $a+b$ as many times as there are units in the index n. And the quotients that ariſe from theſe diviſions are a ſet of infinite ſerieſes conſiſting of terms marked alternately with the ſign — and the ſign +, and of which the numeral coefficients will be the figurate numbers of the ſeveral ſucceſſive orders. This will appear by making a few of theſe diviſions; which I ſhall therefore now proceed to make: but, in order to render the operations ſomewhat ſhorter and eaſier than they otherwiſe would be, I ſhall ſubſtitute the binomial quantity $1+x$ inſtead of the binomial quantity $a+b$, which will make no change whatever in the numeral coefficients of the terms of the ſeveral quotients that will reſult from theſe diviſions: and I ſhall ſuppoſe the quantity x to be leſs than 1, to the end that the powers of x in the terms of the ſeveral quotients may be decreaſing quantities.

 116. The

116. The firſt of theſe diviſions will be as follows:

Diviſor. Quotient.

$1+x)\ (1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11}+$ &c.

Dividend.

$$
\begin{array}{l}
1 \quad * \quad * \quad * \quad * \quad * \quad * \quad * \quad * \\
\underline{1+x} \\
*-x \\
\underline{-x-x^2} \\
*+x^2 \\
\underline{+x^2+x^3} \\
*-x^3 \\
\underline{-x^3-x^4} \\
*+x^4 \\
\underline{+x^4+x^5} \\
*-x^5 \\
\underline{-x^5-x^6} \\
*+x^6 \\
\underline{+x^6+x^7} \\
*-x^7 \\
\underline{-x^7-x^8} \\
*+x^8 \\
\underline{+x^8+x^9} \\
*-x^9 \\
\underline{-x^9-x^{10}} \\
*+x^{10} \\
\underline{+x^{10}+x^{11}} \\
*-x^{11} \\
\underline{-x^{11}-x^{12}} \\
*+x^{12}
\end{array}
$$

By this diviſion it appears that the fraction $\frac{1}{1+x}$ is equal to the infinite ſeries $1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11}+$ &c, in which the ſecond, fourth, ſixth, eighth, tenth, and twelfth terms are marked with the ſign —, or are to be ſubtracted from the firſt term 1; and the third, fifth, ſeventh, ninth, and eleventh terms are marked with the ſign +, or are to be added to the firſt term 1. And it is eaſy to ſee, from the manner of making this diviſion, that, if the operation was to be continued to any greater

greater number of terms whatsoever, the 14th, and 16th, and 18th terms, and all the following even terms in the quotient would also be marked with the sign —; and that the 13th, and 15th, and 17th terms, and all the following odd terms in the quotient would be marked with the sign +. And the numeral co-efficients of all the terms in this quotient are units, or are the terms of the first vertical column of terms in the foregoing table of combinations, or the figurate numbers of the first order; agreeably to what has been just now asserted.

117. The next division will be as follows:

Divisor. Quotient.

$1+x)\ (1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7+9x^8-10x^9+11x^{10}-12x^{11}+ \&c.$

Dividend.

$1-x+x^2-x^3+x^4-x^5+x^6-x^7+x^8-x^9+x^{10}-x^{11}+\&c.$

$\underline{1+x}$

$* -2x+x^2$

$\underline{-2x-2x^2}$

$* +3x^2-x^3$

$\underline{+3x^2+3x^3}$

$* -4x^3+x^4$

$\underline{-4x^3-4x^4}$

$* +5x^4-x^5$

$\underline{+5x^4+5x^5}$

$* -6x^5+x^6$

$\underline{-6x^5-6x^6}$

$* +7x^6-x^7$

$\underline{+7x^6+7x^7}$

$* -8x^7+x^8$

$\underline{-8x^7-8x^8}$

$* +9x^8-x^9$

$\underline{+9x^8+9x^9}$

$* -10x^9+x^{10}$

$\underline{-10x^9-10x^{10}}$

$* +11x^{10}-x^{11}$

$\underline{+11x^{10}+11x^{11}}$

$* -12x^{11}+\&c.$

$\underline{-12x^{11}\quad \&c.}$

$*$

By

By this division it appears that the fraction $\frac{1}{\overline{1+x}|^2}$ is equal to the infinite series $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7 + 9x^8 - 10x^9 + 11x^{10} - 12x^{11} +$ &c, in which, as in the former quotient, the second, fourth, sixth, eighth, tenth, twelfth, and other following even terms have the sign — prefixed to them, or are to be subtracted from the first term 1; and the third, fifth, seventh, ninth, eleventh, and other following odd terms are marked with the sign +, or are to be added to the said first term. And the numeral coefficients of the several terms of this quotient are the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, or the terms of the second vertical column of terms in the foregoing table of combinations, or the figurate numbers of the second order.

118. The

118. The third division will be as follows:

Divisor. Quotient.

$1+x)\ (1-3x+6x^2-10x^3+15x^4-21x^5+28x^6-36x^7+45x^8-55x^9+66x^{10}-78x^{11}+\&c.$

Dividend.

$1-2x+3x^2-4x^3+5x^4-6x^5+7x^6-8x^7+9x^8-10x^9+11x^{10}-12x^{11}+\&c.$

$1+x$

$*-3x+3x^2$

$-3x-3x^2$

$*+6x^2-4x^3$

$+6x^2+6x^3$

$*-10x^3+5x^4$

$-10x^3-10x^4$

$*+15x^4-6x^5$

$+15x^4+15x^5$

$*-21x^5+7x^6$

$-21x^5-21x^6$

$*+28x^6-8x^7$

$+28x^6+28x^7$

$*-36x^7+9x^8$

$-36x^7-36x^8$

$*+45x^8-10x^9$

$+45x^8+45x^9$

$*-55x^9+11x^{10}$

$-55x^9-55x^{10}$

$*+66x^{10}-12x^{11}$

$+66x^{10}+66x^{11}$

$*-78x^{11}+\&c.$

$-78x^{11}-\&c.$

$*\quad+$

By this division it appears that the fraction $\frac{1}{\overline{1+x}|^3}$ is equal to the infinite series $1-3x+6x^2-10x^3+15x^4-21x^5+28x^6-36x^7+45x^8-55x^9+66x^{10}-78x^{11}+\&c$, in which, as in the two former quotients, the second, fourth, sixth, eighth, tenth, twelfth, and other following even terms have the sign — prefixed to them, or are to be subtracted from the first term 1; and the third, fifth, seventh, ninth, eleventh, and other following odd terms have the sign + prefixed to them,

them, or are to be added to the said first term. And the numeral co-efficients of the several terms of this quotient are the terms of the third vertical column in the aforesaid table of combinations, or the figurate numbers of the third order, or the *triangular* numbers.

119. The fourth division will be as follows:

Divisor. Quotient.

$$1+x)\quad(1-4x+10x^2-20x^3+35x^4-56x^5+84x^6-120x^7+165x^8-\&c.$$

Dividend.

$$1-3x+6x^2-10x^3+15x^4-21x^5+28x^6-36x^7+45x^8-\&c.$$
$$\underline{1+x}$$
$$*-4x+6x^2$$
$$\underline{-4x-4x^2}$$
$$*\;+10x^2-10x^3$$
$$\underline{+10x^2+10x^3}$$
$$*\;-20x^3+15x^4$$
$$\underline{-20x^3-20x^4}$$
$$*\;+35x^4-21x^5$$
$$\underline{+35x^4+35x^5}$$
$$*\;-56x^5+28x^6$$
$$\underline{-56x^5-56x^6}$$
$$*\;+84x^6-36x^7$$
$$\underline{+84x^6+84x^7}$$
$$*\;-120x^7+45x^8$$
$$\underline{-120x^7-120x^8}$$
$$*\;+165x^8-\&c.$$
$$\underline{+165x^8+\&c.}$$
$$*\;-\&c.$$

By this division it appears that the fraction $\frac{1}{\overline{1+x}|^4}$ is equal to the infinite series $1-4x+10x^2-20x^3+35x^4-56x$ $+84x^6-120x^7+165x^8-$ &c, *ad infinitum*, in which, as in the three former quotients, the second, fourth, sixth, and eighth, and other following even terms have the sign — prefixed to them, or are to be subtracted from the first term 1; and the third, fifth, seventh, ninth, and other following odd terms have the sign + prefixed to them, or are to be added

added to the said first term. And the numeral co-efficients of the several terms of this quotient are the terms of the fourth vertical column in the aforesaid table of combinations, or the figurate numbers of the fourth order, or the *pyramidal* numbers.

120. The fifth division will be as follows:

Divisor. Quotient.

$$1+x)\ (1-5x+15x^2-35x^3+70x^4-126x^5+210x^6-\&c.$$

Dividend.

$$1-4x+10x^2-20x^3+35x^4-56x^5+84x^6-120x^7+\&c.$$
$$\underline{1+x}$$
$$*-5x+10x^2$$
$$\underline{-5x-5x^2}$$
$$*+15x^2-20x^3$$
$$\underline{+15x^2+15x^3}$$
$$*-35x^3+35x^4$$
$$\underline{-35x^3-35x^4}$$
$$*+70x^4-56x^5$$
$$\underline{+70x^4+70x^5}$$
$$*-126x^5+84x^6$$
$$\underline{-126x^5-126x^6}$$
$$*+210x^6-\&c.$$
$$\underline{+210x^6+\&c.}$$
$$*-\&c.$$

By this division it appears that the fraction $\frac{1}{\overline{1+x}|^5}$ is equal to the infinite series $1-5x+15x^2-35x^3+70x^4-126x^5+210x^6-\&c$, *ad infinitum*, in which, as in the four preceeding quotients, the second, fourth, sixth, and other following even terms have the sign $-$ prefixed to them, or are to be subtracted from the first term 1; and the third, fifth, seventh, and other following odd terms have the sign $+$ prefixed to them, or are to be added to the said first term. And the numeral co-efficients of the several terms of this quotient are the terms of the fifth vertical column in the foregoing table of combinations, or the figurate numbers of the fifth order.

121. The sixth division will be as follows:

Divisor. Quotient.

$1+x$) $(1-6x+21x^2-56x^3+126x^4-252x^5+$ &c.

Dividend.

$$\begin{array}{l} 1-5x+15x^2-35x^3+70x^4-126x^5+\&c. \\ \underline{1+x} \\ *-6x+15x^2 \\ \quad \underline{-6x-\ 6x^2} \\ \quad\quad *+21x^2-35x^3 \\ \quad\quad\quad \underline{+21x^2+21x^3} \\ \quad\quad\quad\quad *-56x^3+70x^4 \\ \quad\quad\quad\quad\quad \underline{-56x^3-56x^4} \\ \quad\quad\quad\quad\quad\quad *+126x^4-126x^5 \\ \quad\quad\quad\quad\quad\quad\quad \underline{+126x^4+126x^5} \\ \quad\quad\quad\quad\quad\quad\quad\quad *-252x^5+\&c. \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{-252x^5-\&c.} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad *\ +\&c. \end{array}$$

By this division it appears that the fraction $\frac{1}{\overline{1+x}|^6}$ is equal to the infinite series $1 - 6x + 21x^2 - 56x^3 + 126x^4 - 252x^5 +$ &c; in which, as in the five former quotients, the second, fourth, sixth, and other following even terms have the sign $-$ prefixed to them, or are to be subtracted from the first term 1; and the third, and fifth, and other following odd terms have the sign $+$ prefixed to them, or are to be added to the said first term. And the numeral co-efficients of the several terms of this quotient are the terms of the sixth vertical column in the foregoing table of combinations, or the figurate numbers of the sixth order.

Conclusions

Conclusions from the foregoing Operations of Division.

122. From the operations of the foregoing six divisions with the same divisor $1 + x$, I presume that it will be evident to the reader, that, if we were to continue each of the foregoing quotients to any number of terms, how great soever, the said terms would continue to be marked with the signs + and — alternately, and that the co-efficients of the following terms after those that have been above computed, would be the following numbers of the same order of figurate numbers to which the co-efficients of the terms above computed in the said quotients, belonged, respectively. And I likewise presume that it will be evident to him, that, if we were to divide the last, or sixth, quotient by $1 + x$, and the next, or seventh, quotient, by the same quantity $1 + x$, and the several next following, or the eighth, ninth, tenth, and eleventh, &c, quotients, continued to any number whatsoever, by the same quantity $1 + x$ (whereby we should obtain several infinite serieses that would be equal to the fractions $\frac{1}{\overline{1+x}|^7}$, $\frac{1}{\overline{1+x}|^8}$, $\frac{1}{\overline{1+x}|^9}$, $\frac{1}{\overline{1+x}|^{10}}$, $\frac{1}{\overline{1+x}|^{11}}$, &c), and were to continue each of the said divisions till we had obtained any number of terms in the quotient, how great soever, the several even terms in every quotient would be marked with the sign —, or subtracted from the first term 1; and the third, fifth, seventh, and other following odd terms in every quotient would be marked with the sign +, or added to the said first term; and also, that the co-efficients of the terms of the said 7th quotient (that would be equal to the fraction $\frac{1}{\overline{1+x}|^7}$) would be the figurate numbers of the 7th order, and the co-efficients of the terms of the said 8th

 quotient

quotient (that would be equal to the fraction $\frac{1}{\overline{1+x}|^{8}}$,) would be the figurate numbers of the 8th order, and that the coefficients of the terms of the 9th, 10th, 11th, and other following quotients (which would be equal to the fractions $\frac{1}{\overline{1+x}|^{9}}$, $\frac{1}{\overline{1+x}|^{10}}$, $\frac{1}{\overline{1+x}|^{11}}$, &c) would be the figurate numbers of the 9th, 10th, 11th, and other following orders, respectively.

Obſervations on the foregoing Operations of Diviſion, tending to eſtabliſh the foregoing Concluſions.

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123. The foregoing concluſions may be derived from the following obſervations, which cannot but occur to every perſon who ſhall go through the foregoing operations of algebraïck diviſion with attention, namely,

1ſt, That in every ſeparate operation of diviſion, by which a new term in the quotient is to be obtained, the dividend will always conſiſt of two terms which will have different ſigns + and — prefixed to them; ſo that, when the firſt of the two has the ſign + prefixed to it, the ſecond will be marked with the ſign —; and when the firſt has the ſign — prefixed to it, the ſecond will be marked with the ſign +.

2dly, That the ſubtrahend, or quantity which is to be ſubtracted from the ſaid dividend, will always conſiſt of two terms, which will be both marked with the ſame ſign + or —, which ſign will alſo be the ſame with that of the firſt of the two terms of the dividend from which the ſaid ſubtrahend is to be ſubtracted; and therefore the ſign which is prefixed

prefixed to the ſecond term of the ſaid ſubtrahend will be contrary to that which is prefixed to the ſecond term of the ſaid dividend, from which it is to be ſubtracted; whence it follows that when, in order to ſubtract the ſaid ſecond term of the ſubtrahend from the ſecond term of the dividend, which is placed juſt above it, we ſhall (according to the rules of algebraïck ſubtraction) have changed its ſign into the contrary ſign, and have added it, with its ſign ſo changed, to the ſecond term of the dividend, the reſidue thence reſulting (which will be the firſt term of the next dividend) will have the ſame ſign prefixed to it as is prefixed to the ſecond term of the former dividend, or the contrary ſign to that which is prefixed to the firſt term of the former dividend; ſo that the firſt terms of every two contiguous dividends throughout the whole diviſion will be marked with contrary ſigns, and conſequently every two contiguous terms in the quotient (which have always the ſame ſigns with the firſt terms of the two dividends from which they are derived) will alſo be marked with contrary ſigns.

3dly, Since the two terms of the diviſor $1 + x$, to wit, 1 and x, have the ſame numeral co-efficient 1, and every new ſubtrahend is produced by multiplying the diviſor $1 + x$ into the laſt-found term of the quotient, it follows that the numeral co-efficient of the ſecond term of every new ſubtrahend muſt be the ſame with the numeral co-efficient of the firſt term of the ſame ſubtrahend. And conſequently, when the ſign of the ſecond term of the ſubtrahend is changed, and thereby become the ſame with the ſign of the ſecond term of the dividend, which is juſt above it, and it is added, with its ſign ſo changed, to the ſaid ſecond term of the dividend, the co-efficient of the quantity reſulting from this addition, or algebraïcal ſubtraction (which is evidently the ſum of the co-efficient of the ſecond term of the ſubtrahend and of the co-efficient of the ſecond term of the dividend) will alſo be the ſum of the co-efficient of the firſt term of the ſubtrahend and of the co-efficient of the ſecond term of the dividend, and conſequently (becauſe the firſt term of the ſubtrahend is always equal to, or the ſame

ſame with, the firſt term of the dividend) will alſo be the ſum of the co-efficient of the firſt term of the dividend and the co-efficient of the ſecond term of the dividend; that is, the co-efficient of the firſt term of every new dividend will be the ſum of the co-efficients of the firſt and ſecond terms of the next preceeding dividend. And conſequently the co-efficient of every new term in the quotient (which is the ſame with the co-efficient of the firſt term of the dividend from which it is derived) will be the ſum of the two co-efficients of the two terms of the next preceeding dividend. But the ſecond term of the next preceeding dividend is a term of the laſt preceeding ſeries, or quotient obtained by the diviſion by $1 + x$; and the co-efficient of the firſt term of the ſaid next preceeding dividend is equal to the ſum of the co-efficients of all the preceeding terms of the ſaid laſt preceeding ſeries, or quotient obtained by the diviſion by $1 + x$. Therefore the co-efficient of every new term in the quotient ariſing from the preſent diviſion by $1 + x$ will be equal to the ſum of all the co-efficients of the terms in the foregoing ſeries, or quotient, as far as the correſponding term, or term involving the ſame power of x, and including the ſaid term. Thus, if the former ſeries, or quotient, be called A, and the preſent quotient, now ariſing from the diviſion of the ſeries A by $1 + x$, be called B, and m be a whole number denoting the place of any term in the quotient B, the co-efficient of the mth term of the ſeries, or quotient, B, will be equal to the ſum of the co-efficients of all the terms of the preceeding ſeries, or quotient, A, as far as the mth term of the ſaid ſeries, and including the ſaid mth term.

And therefore, 4thly, that the co-efficients of the terms of the ſeveral ſerieſes, or quotients, ariſing by the continual diviſion of 1 by the binomial quantity $1 + x$ will be the ſeveral orders of figurate numbers, or the terms of the ſeveral vertical columns of terms in the foregoing table of combinations; ſince both the ſaid co-efficients of the terms of the ſaid ſerieſes, or quotients, and the ſaid figurate numbers, or terms of the ſeveral vertical columns of terms in the ſaid faid table of combinations, arife in the fame manner from a feries of units, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c, to wit, by the continual addition of them to each other, and by the like continual addition of the terms of every feries generated from them to each other.

Application of the foregoing reafonings to the finding of a general expreffion of the value of the fraction $\frac{1}{\overline{a+b}|^n}$ *in an infinite feries of fimple terms.*

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124. It having been now proved that the terms of the feveral feriefes, or quotients, that are equal to the fractions $\frac{1}{1+x}$, $\frac{1}{\overline{1+x}|^2}$, $\frac{1}{\overline{1+x}|^3}$, $\frac{1}{\overline{1+x}|^4}$, $\frac{1}{\overline{1+x}|^5}$, $\frac{1}{\overline{1+x}|^6}$, &c, *ad infinitum*, (beginning with the fecond term in each feries), are to be marked with the fign — and the fign + alternately, and that the co-efficients of the terms of the faid feriefes will be the figurate numbers of the correfponding orders; and it having been proved above in coroll. 10, that the figurate numbers of the *n*th order, or the fignificant terms of the *n*th vertical column of terms in the foregoing table of combinations (*n* being put for any whole number whatfoever) are equal to the terms of the following feries, to wit, 1, $\frac{n}{1}$ A, $\frac{n+1}{2}$ B, $\frac{n+2}{3}$ C, $\frac{n+3}{4}$ D, $\frac{n+4}{5}$ E, $\frac{n+5}{6}$ F, $\frac{n+6}{7}$ G, $\frac{n+7}{8}$ H, $\frac{n+8}{9}$ I, $\frac{n+9}{10}$ K, $\frac{n+10}{11}$ L, &c, *ad infinitum*; it follows that the fraction $\frac{1}{\overline{1+x}|^n}$ will be equal to the infinite feries $1 - \frac{n}{1} Ax$

+

$+ \frac{n+1}{2} B x^2 - \left|\overline{\frac{n+2}{3}}\right. C x^3 + \frac{n+3}{4} D x^4 - \left|\overline{\frac{n+4}{5}}\right. E x^5 + \frac{n+5}{6}$ $F x^6 - \left|\overline{\frac{n+6}{7}}\right. G x^7 + \frac{n+7}{8} H x^8 - \left|\overline{\frac{n+8}{9}}\right. I x^9 + \frac{n+9}{10} K x^{10} -$ $\left|\overline{\frac{n+10}{11}}\right. L x^{11}$ + &c, *ad infinitum.*

125. Now let $\frac{b}{a}$ be substituted instead of x. And the fraction $\frac{1}{1+x}$ will then be $= \frac{1}{1+\frac{b}{a}}$, and the series $1 - \frac{n}{1} A x$ $+ \frac{n+1}{2} B x^2 - \left|\overline{\frac{n+2}{3}}\right. C x^3 + \frac{n+3}{4} D x^4 - \left|\overline{\frac{n+4}{5}}\right. E x^5 + \frac{n+5}{6}$ $F x^6 - \left|\overline{\frac{n+6}{7}}\right. G x^7 + \frac{n+7}{8} H x^8 - \left|\overline{\frac{n+8}{9}}\right. I x^9 + \frac{n+9}{10}$ $K x^{10} - \left|\overline{\frac{n+10}{11}}\right. L x^{11}$ &c, *ad infinitum*, will be $= 1 - \frac{n}{1} A$ $\frac{b}{a} + \frac{n+1}{2} B \frac{b^2}{a^2} - \left|\overline{\frac{n+2}{3}}\right. C \frac{b^3}{a^3} + \frac{n+3}{4} D \frac{b^4}{a^4} - \left|\overline{\frac{n+4}{5}}\right. E \frac{b^5}{a^5} +$ $\frac{n+5}{6} F \frac{b^6}{a^6} - \left|\overline{\frac{n+6}{7}}\right. G \frac{b^7}{a^7} + \frac{n+7}{8} H \frac{b^8}{a^8} - \left|\overline{\frac{n+8}{9}}\right. I \frac{b^9}{a^9} + \frac{n+9}{10}$ $K \frac{b^{10}}{a^{10}} - \left|\overline{\frac{n+10}{11}}\right. L \frac{b^{11}}{a^{11}}$ + &c, *ad infinitum.* Therefore the fraction $\frac{1}{\left.\overline{1+\frac{b}{a}}\right|^n}$ will be equal to the series $1 - \frac{n}{1} A \frac{b}{a} +$ $\frac{n+1}{2} B \frac{b^2}{a^2} - \left|\overline{\frac{n+2}{3}}\right. C \frac{b^3}{a^3} + \frac{n+3}{4} D \frac{b^4}{a^4} - \left|\overline{\frac{n+4}{5}}\right. E \frac{b^5}{a^5} + \frac{n+5}{6}$ $F \frac{b^6}{a^6} - \left|\overline{\frac{n+6}{7}}\right. G \frac{b^7}{a^7} + \frac{n+7}{8} H \frac{b^8}{a^8} - \left|\overline{\frac{n+8}{9}}\right. I \frac{b^9}{a^9} + \frac{n+9}{10} K \frac{b^{10}}{a^{10}}$ $- \left|\overline{\frac{n+10}{11}}\right. L \frac{b^{11}}{a^{11}}$ + &c; and consequently (dividing both sides of this equation by a^n) we shall have ($\frac{\frac{1}{\left.\overline{1+\frac{b}{a}}\right|^n}}{a^n}$,

or

or $\frac{1}{\overline{1+\frac{b}{a}}|^{n} \times a^{n}}$, or $\frac{1}{\overline{1+\frac{b}{a}\times a}|^{n}}$, or) $\frac{1}{\overline{a+b}|^{n}}$, $= \frac{1}{a^{n}} - \frac{n}{1}\,A\,\frac{b}{a^{n+1}} + \frac{n+1}{2}\,B\,\frac{b^{2}}{a^{n+2}} - \frac{n+2}{3}\,C\,\frac{b^{3}}{a^{n+3}} + \frac{n+3}{4}\,D\,\frac{b^{4}}{a^{n+4}} - \frac{n+4}{5}\,E\,\frac{b^{5}}{a^{n+5}} + \frac{n+5}{6}\,F\,\frac{b^{6}}{a^{n+6}} - \frac{n+6}{7}\,G\,\frac{b^{7}}{a^{n+7}} + \frac{n+7}{8}\,H\,\frac{b^{8}}{a^{n+8}} - \frac{n+8}{9}\,I\,\frac{b^{9}}{a^{n+9}} + \frac{n+9}{10}\,K\,\frac{b^{10}}{a^{n+10}} - \frac{n+10}{11}\,L\,\frac{b^{11}}{a^{n+11}}$ + &c, *ad infinitum*; or, according to Sir Isaac Newton's notation with negative indexes of powers, we shall have $\overline{a+b}|^{-n}$ = the series $a^{-n} - \frac{n}{1}\,A\,a^{-n-1}\,b + \frac{n+1}{2}\,B\,a^{-n-2}\,b^{2} - \frac{n+2}{3}\,C\,a^{-n-3}\,b^{3} + \frac{n+3}{4}\,D\,a^{-n-4}\,b^{4} - \frac{n+4}{5}\,E\,a^{-n-5}\,b^{5} + \frac{n+5}{6}\,F\,a^{-n-6}\,b^{6} - \frac{n+6}{7}\,G\,a^{-n-7}\,b^{7} + \frac{n+7}{8}\,H\,a^{-n-8}\,b^{8} - \frac{n+8}{9}\,I\,a^{-n-9}\,b^{9} + \frac{n+9}{10}\,K\,a^{-n-10}\,b^{10} - \frac{n+10}{11}\,L\,a^{-n-11}\,b^{11}$ + &c, *ad infinitum*. Q. E. I.

126. This last series is the same with that which would result from Sir Isaac Newton's original series for expressing the value of the quantity $\overline{a+b}|^{n}$, or the nth power of the binomial quantity $a+b$, to wit, the series $a^{n} + \frac{n}{1}\,a^{n-1}\,b + \frac{n}{1} \times \frac{n-1}{2}\,a^{n-2}\,b^{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}\,a^{n-3}\,b^{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}\,a^{n-4}\,b^{4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}\,a^{n-5}\,b^{5}$ + &c, by supposing the index n of the said power to be negative, or by substituting $-n$ instead of n in the terms of the said series.

 For,

For, if this substitution be made in the terms of the said series, it will become equal to $a^{-n} - \frac{n}{1} a^{-n-1} b^{1} - \frac{n}{1} \times \frac{-n-1}{2} a^{-n-2} b^{2} - \frac{n}{1} \times \frac{-n-1}{2} \times \frac{-n-2}{3} a^{-n-3} b^{3} - \frac{n}{1} + \frac{-n-1}{2} \times \frac{-n-2}{3} \times \frac{-n-3}{4} \times a^{-n-4} b^{4} - \frac{n}{1} \times \frac{-n-1}{2} \times \frac{-n-2}{3} \times \frac{-n-3}{4} \times \frac{-n-4}{5} \times a^{-n-5} b^{5} -$ &c, and consequently, (because $-\frac{n}{1} \times \frac{-n-1}{2}$ is $= +\frac{n}{1} \times \frac{+n+1}{2}$, and $\frac{-n-2}{3} \times \frac{-n-3}{4}$ is $= \frac{+n+2}{3} \times \frac{+n+3}{4}$), equal to $a^{-n} - \frac{n}{1} a^{-n-1} b^{1} + \frac{n}{1} \times \frac{+n+1}{2} a^{-n-2} b^{2} + \frac{n}{1} \times \frac{+n+1}{2} \times \frac{-n-2}{3} a^{-n-3} b^{3} + \frac{n}{1} \times \frac{+n+1}{2} \times \frac{+n+2}{3} \times \frac{+n+3}{4} a^{-n-4} b^{4} + \frac{n}{1} \times \frac{+n+1}{2} \times \frac{+n+2}{3} \times \frac{+n+3}{4} \times \frac{-n-4}{5} a^{-n-5} b^{5} -$ &c, $= a^{-n} - \frac{n}{1} a^{-n-1} b^{1} + \frac{n}{1} \times \frac{n+1}{2} a^{-n-2} b^{2} + \frac{n}{1} \times \frac{n+1}{2} \times -\frac{n+2}{3} a^{-n-3} b^{3} + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} a^{-n-4} b^{4} + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times -\frac{n+4}{5} a^{-n-5} b^{5} +$ &c, $= a^{-n} - \frac{n}{1} A a^{-n-1} b^{1} + \frac{n+1}{2} B a^{-n-2} b^{2} - \frac{n+2}{3} C a^{-n-3} b^{3} + \frac{n+3}{4} D a^{-n-4} b^{4} - \frac{n+4}{5} E a^{-n-5} b^{5} +$ &c; which is the series we just now derived from Mr. James Bernoulli's doctrine of combinations for the value of the quantity $\overline{a+b}|^{-n}$, or $\frac{1}{\overline{a+b}|^{n}}$. Q. E. D.

A general

A general expression of the value of the fraction $\frac{1}{\overline{a-b}|^n}$, or of the reciprocal of any integral power of the residual quantity $a-b$, in an infinite series.

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127. Coroll. 12. The fraction $\frac{1}{\overline{a-b}|^n}$ will be equal to the series $\frac{1}{a^n} + \frac{n}{1} A \frac{b}{a^{n+1}} + \frac{n+1}{2} B \frac{b^2}{a^{n+2}} + \frac{n+2}{3} C \frac{b^3}{a^{n+3}} + \frac{n+3}{4} D \frac{b^4}{a^{n+4}} + \frac{n+4}{5} E \frac{b^5}{a^{n+5}} + \frac{n+5}{6} F \frac{b^6}{a^{n+6}} + \frac{n+6}{7} G \frac{b^7}{a^{n+7}} + \frac{n+7}{8} H \frac{b^8}{a^{n+8}} + \frac{n+8}{9} I \frac{b^9}{a^{n+9}} + \frac{n+9}{10} K \frac{b^{10}}{a^{n+10}} + \frac{n+10}{11} L \frac{b^{11}}{a^{n+11}} +$ &c, *ad infinitum*, which consists of the very same terms as the series obtained in the foregoing corollary for the value of the fraction $\frac{1}{\overline{a+b}|^n}$, but with the sign $+$ prefixed to all the terms after the first term $\frac{1}{a^n}$, instead of being prefixed only to the 3d, 5th, 7th, 9th, and other following odd terms of it, as in the former series.

128. This will appear by dividing 1 two or three times following by the residual quantity $1-x$ instead of the binomial quantity $1+x$. For we shall easily perceive that all the terms, after the first terms, in the quotients arising from those divisions will be marked with the sign $+$, or must be added to the first terms. The three first of these divisions will be as follows:

 The

The first Division.

Divisor. Quotient.

$1-x)\ (1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+$ &c.

Dividend.

$1\ *\ *\ *\ *\ *\ *\ *\ *\ *\ *\ *$

$1-x$

$+x\ *$

$+x-x^2$

$*+x^2\ *$

$+x^2-x^3$

$*+x^3\ *$

$+x^3-x^4$

$*+x^4\ *$

$+x^4-x^5$

$*+x^5\ *$

$+x^5-x^6$

$*+x^6\ *$

$+x^6-x^7$

$*+x^7\ *$

$+x^7-x^8$

$*+x^8\ *$

$+x^8-x^9$

$*+x^9\ *$

$+x^9-x^{10}$

$*+x^{10}$

The second Division.

Divisor. Quotient.

$1-x)\ (1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+9x^8+10x^9+$ &c.

Dividend.

$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+$ &c.

$\underline{1-x}$

$+2x+x^2$

$\underline{+2x-2x^2}$

$*\ +3x^2+x^3$

$\underline{+3x^2-3x^3}$

$*\ +4x^3+x^4$

$\underline{+4x^3-4x^4}$

$*\ +5x^4+x^5$

$\underline{+5x^4-5x^5}$

$*\ +6x^5+x^6$

$\underline{+6x^5-6x^6}$

$*\ +7x^6+x^7$

$\underline{+7x^6-7x^7}$

$*\ +8x^7+x^3$

$\underline{+8x^7-8x^8}$

$*\ +9x^8$

$\underline{+9x^8}$

$*$

The third Division.

Divisor. Quotient.

$1-x)\quad(1+3x+6x^2+10x^3+15x^4 21x^5+36x^6+\&c.$

Dividend.

$$1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+9x^8+\&c.$$
$$\underline{1-x}$$
$$+3x+3x^2$$
$$\underline{+3x-3x^2}$$
$$*\ +6x^2+4x^3$$
$$\underline{+6x^2-6x^3}$$
$$*\ +10x^3+5x^4$$
$$\underline{+10x^3-10x^4}$$
$$*\ +15x^4+6x^5$$
$$\underline{+15x^4-15x^5}$$
$$*\ +21x^5+7x^6$$
$$\underline{+21x^5-21x^6}$$
$$*\ +28x^6+8x^7$$
$$\underline{+28x^6-28x^7}$$
$$*\ +36x^7$$

129. It is easy to see that, both in these three divisions, and in all the following divisions that may be made of the last quotient hereby obtained, by the same divisor $1-x$, all the terms of the several quotients, after the first terms, will be marked with the sign $+$, or be added to the first terms, and that the co-efficients of the several terms will be the very same numbers as the co-efficients of the corresponding terms in the former quotients which resulted from the divisions by the binomial quantity $1+x$. It follows therefore that the fraction $\frac{1}{\overline{1-x}|^{n}}$ will be equal to the infinite series $1+\frac{n}{1}Ax+\frac{n+1}{2}Bx^2+\frac{n+2}{3}Cx^3+\frac{n+3}{4}Dx^4+\frac{n+4}{5}Ex^5+\frac{n+5}{6}Fx^6+\frac{n+6}{7}Gx^7+\frac{n+7}{8}Hx^8+\frac{n+8}{9}Ix^9+\frac{n+9}{10}Kx^{10}+\frac{n+10}{11}Lx^{11}+$ &c, *ad infinitum*, and consequently

quently (ſubſtituting $\frac{b}{a}$ inſtead of x in this equation) that the fraction $\frac{1}{\overline{1-\frac{b}{a}}|^n}$ will be equal to the infinite ſeries $1 + \frac{n}{1} A \frac{b}{a} + \frac{n+1}{2} B \frac{b^2}{a^2} + \frac{n+2}{3} C \frac{b^3}{a^3} + \frac{n+3}{4} D \frac{b^4}{a^4} + \frac{n+4}{5} E \frac{b^5}{a^5} + \frac{n+5}{6} F \frac{b^6}{a^6} + \frac{n+6}{7} G \frac{b^7}{a^7} + \frac{n+7}{8} H \frac{b^8}{a^8} + \frac{n+8}{9} I \frac{b^9}{a^9} + \frac{n+9}{10} K \frac{b^{10}}{a^{10}} + \frac{n+10}{11} L \frac{b^{11}}{a^{11}} +$ &c, *ad infinitum*, and conſequently (dividing both ſides of the laſt equation by a^n) that the fraction ($\frac{\frac{1}{\overline{1-\frac{b}{a}}|^n}}{a^n}$, or $\frac{1}{a^n \times \overline{1-\frac{b}{a}}|^n}$, or $\frac{1}{\overline{a \times 1-\frac{b}{a}}|^n}$, or) $\frac{1}{\overline{a-b}|^n}$ will be equal to the infinite ſeries $\frac{1}{a^n} + \frac{n}{1} A \frac{b}{a^{n+1}} + \frac{n+1}{2} B \frac{b^2}{a^{n+2}} + \frac{n+2}{3} C \frac{b^3}{a^{n+3}} + \frac{n+3}{4} D \frac{b^4}{a^{n+4}} + \frac{n+4}{5} E \frac{b^5}{a^{n+5}} + \frac{n+5}{6} F \frac{b^6}{a^{n+6}} + \frac{n+6}{7} G \frac{b^7}{a^{n+7}} + \frac{n+7}{8} H \frac{b^8}{a^{n+8}} + \frac{n+8}{9} I \frac{b^9}{a^{n+9}} + \frac{n+9}{10} K \frac{b^{10}}{a^{n+10}} + \frac{n+10}{11} L \frac{b^{11}}{a^{n+11}} +$ &c, *ad infinitum*. Q. E. D.

130. If we make uſe of Sir Iſaac Newton's notation with negative indexes of powers, the laſt equation will be as follows, to wit, $\overline{a-b}|^{-n} =$ the infinite ſeries $a^{-n} + \frac{n}{1} A a^{-n-1} b + \frac{n+1}{2} B a^{-n-2} b^2 + \frac{n+2}{3} C a^{-n-3} b^3 + \frac{n+3}{4} D a^{-n-4} b^4 + \frac{n+4}{5} E a^{-n-5} b^5 + \frac{n+5}{6} F a^{-n-6} b^6 + \frac{n+6}{7}$

$G a^{-n-7} b^7 + \frac{n+7}{8} H a^{-n-8} b^8 + \frac{n+8}{9} I a^{-n-9} b^9 + \frac{n+9}{10} K a^{-n-10} b^{10} + \frac{n+10}{11} L a^{-n-11} b^{11}$ + &c, *ad infinitum.* But the other way of expressing this equation seems to be clearer and more natural than this way, and, for ordinary purposes, preferable to it.

131. This last series is the same with that which would result from Sir Isaac Newton's original series for expressing the value of the quantity $\overline{a-b}|^n$, or the nth power of the residual quantity $a-b$, to wit, the series $a^n - \frac{n}{1} a^{n-1} b^1 + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} b^2 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} b^4 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} a^{n-5} b^5$ + &c, by supposing the index n of the said power to be negative, or by substituting $-n$ instead of n in the terms of the said series.

For, if this substitution be made in the terms of the said series, it will become equal to $a^{-n} + \frac{n}{1} a^{-n-1} b^1 - \frac{n}{1} \times \frac{-n-1}{2} a^{-n-2} b^2 + \frac{n}{1} \times \frac{-n-1}{2} \times \frac{-n-2}{3} a^{-n-3} b^3 - \frac{n}{1} \times \frac{-n-1}{2} \times \frac{-n-2}{3} \times \frac{-n-3}{4} a^{-n-4} b^4 + \frac{n}{1} \times \frac{-n-1}{2} \times \frac{-n-2}{3} \times \frac{-n-3}{4} \times \frac{-n-4}{5} a^{-n-5} b^5 + \&c, = a^{-n} + \frac{n}{1} a^{-n-1} b^1 + \frac{n}{1} \times \frac{+n+1}{2} a^{-n-2} b^2 + \frac{n}{1} \times \frac{+n+1}{2} \times \frac{+n+2}{3} a^{-n-3} b^3 + \frac{n}{1} \times \frac{+n+1}{2} \times \frac{+n+2}{3} \times \frac{+n+3}{4} a^{-n-4} b^4 + \frac{n}{1} \times \frac{+n+1}{2} \times \frac{+n+2}{3} \times \frac{+n+3}{4} \times \frac{+n+4}{5} a^{-n-5} b^5 + \&c, = a^{-n} + \frac{n}{1}$

a^{-n-1}

$a^{-n-1}b^1 + \frac{n}{1} \times \frac{n+1}{2} a^{-n-2} b^2 + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$
$a^{-n-3} b^3 + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} a^{-n-4} b^4 + \frac{n}{1}$
$\times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} a^{-n-5} b^5 +$ &c, $= a^{-n}$
$+ \frac{n}{1} A a^{-n-1} b^1 + \frac{n+1}{2} B a^{-n-2} b^2 + \frac{n+2}{3} C a^{-n-3} b^3$
$+ \frac{n+3}{4} D a^{-n-4} b^4 + \frac{n+4}{5} E a^{-n-5} b^5 +$ &c; which is the series we just now derived in art. 130, from Mr. James Bernoulli's doctrine of combinations for the value of the quantity $\overline{a-b}|^{-n}$, or $\frac{1}{\overline{a-b}|^n}$. Q. E. D.

132. We have now seen how from Mr. James Bernoulli's doctrine of combinations, and his explanation of the properties of the figurate numbers derived from it, may be deduced just and regular demonstrations of Sir Isaac Newton's famous binomial and residual theorems in the case of the integral and negative powers of a binomial and a residual quantity, or of the reciprocals of their integral and affirmative powers, as well as in the case of their integral and affirmative powers themselves, in which Mr. Bernoulli himself has demonstrated them above in coroll. 5. And I doubt whether any other method of demonstrating these two famous theorems in the case of the integral and negative powers of a binomial and a residual quantity has yet been found out, that is equally clear and satisfactory.

A Difficulty that may occur concerning the foregoing Theorem relating to the integral and negative Powers of a residual Quantity, as a — b, *or* 1 — x.

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133. Before we conclude this subject of the binomial and residual theorems in the case of integral and negative powers, I will endeavour to clear up a difficulty which may, perhaps, occur to the reader's mind concerning the latter of the said theorems.

It has been shewn in art. 129, that $\overline{1-x}|^{-n}$, or $\frac{1}{\overline{1-x}|^{n}}$, is equal to the infinite series $1 + \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 + \frac{n+2}{3} Cx^3 + \frac{n+3}{4} Dx^4 + \frac{n+4}{5} Ex^5 + \frac{n+5}{6} Fx^6 + \frac{n+6}{7} Gx^7 +$ &c, *ad infinitum*, in which all the terms following the first term 1 are marked with the sign +, or are to be added to the said first term. And the co-efficients of the terms in this series continually increase, when n is of any magnitude greater than 1. Thus, if n is $= 2$, we shall have $\frac{n}{1}$ $(= \frac{2}{1})$ $= 2$, and $\frac{n+1}{2}$ $(= \frac{2+1}{2}) = \frac{3}{2}$, and $\frac{n+2}{3}$ $(= \frac{2+2}{3}) = \frac{4}{3}$, and $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, equal to $\frac{5}{4}$, $\frac{6}{5}$, $\frac{7}{6}$, $\frac{8}{7}$, &c, respectively; in all which fractions the numerators exceed the denominators by an unit; and, if n is $= 3$, we shall have $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, equal to $\frac{3}{1}$, $\frac{4}{2}$, $\frac{5}{3}$, $\frac{6}{4}$, $\frac{7}{5}$, $\frac{8}{6}$, $\frac{9}{7}$, &c, respectively; in all which fractions the numerators exceed the denominators by 2. And the like excess of the numerators above the denomi-

 nators

nators will take place in a ſtill higher degree in the ſaid generating fractions $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, when the index n is equal to 4, or 5, or 6, or any higher number. And conſequently the co-efficients B, C, D, E, F, G, &c, which are derived from the firſt term 1, or A, by the continual multiplication of the ſaid generating fractions $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, muſt continually increaſe in all theſe ſerieſes. And accordingly we find that the figurate numbers of every order, or the ſeveral ſignificant terms in every vertical column of terms in the foregoing table of combinations, page 74, (which are equal to the co-efficients of the terms of the foregoing ſeries $1 + \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 + \frac{n+2}{3} Cx^3 + \frac{n+3}{4} Dx^4 + \frac{n+4}{5} Ex^5 + \frac{n+5}{6} Fx^6 +$ &c, *ad infinitum*), increaſe continually. And hence it may happen that, if x is but little leſs than 1, the whole terms at the beginning of the ſeries $1 + \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 + \frac{n+2}{3} Cx^3 + \frac{n+3}{4} Dx^4 + \frac{n+4}{5} Ex^5 + \frac{n+5}{6} Fx^6 +$ &c, *ad infinitum*, may (by means of this increaſe of their co-efficients) be increaſing quantities.

Now, from this circumſtance it may, perhaps, be apprehended, that all the terms of this ſeries will in ſome caſes diverge, or increaſe, continually, to what number of terms ſoever the ſaid ſeries may be continued, and conſequently that the ſaid ſeries (conſiſting of an infinite number of terms that are every one greater than that next before it) will be infinite in magnitude as well as in the number of its terms, and therefore cannot be equal to the finite quantity $\frac{1}{\overline{1-x}|^n}$. This is a difficulty that ſeems naturally to ariſe upon this ſubject. But it may be removed by the following conſiderations.

An Explanation of the foregoing Difficulty.

134. The proportion of the numerators of the several generating fractions $\frac{n}{1}$, $\frac{n+1}{2}$, $\frac{n+2}{3}$, $\frac{n+3}{4}$, $\frac{n+4}{5}$, $\frac{n+5}{6}$, $\frac{n+6}{7}$, &c, to their denominators (though it is always a proportion of majority, when n is greater than 1) approaches continually nearer and nearer to a ratio of equality, as its limit; so that, if n be ever so great a number, we may, by continuing the series of these generating fractions to a great number of terms, come to one in which the ratio of the numerator to the denominator shall be less than any proposed ratio of majority. Thus, for example, if n is $=$ 1000, and the ratio of majority that is proposed, or given, and with which the ratios of the numerators of these generating fractions to their denominators is to be compared, is that of 1 to 0.99999, or of 100,000 to 99,999, it will be possible, by continuing the series of the said generating fractions, to assign one in which the ratio of the numerator to the denominator shall be less than the ratio of 1 to 0.99999, or of 100,000 to 99,999. This may be shewn in the manner following. Let z be the general representative of the several numbers added to the index n in the numerators of these successive generating fractions; so that the said numerators shall be equal to the several successive values of $n+z$, or, on the present supposition that n is $=$ 1000, to the several successive values of $1000 + z$. Then will the denominators of the said successive generating fractions be denoted by the successive values of $z + 1$, and the said generating fractions themselves will be equal to the several successive values of the fraction $\frac{1000+z}{z+1}$, or $\frac{z+1000}{z+1}$. Now it is evident that, by continually increasing the number z, the proportion of the numerator $z + 1000$ to the denominator $z + 1$ may be made to approach

proach as near as we pleaſe to the proportion of z to z, or the proportion of equality. The number z may therefore be increaſed till the ſaid proportion of $z + 1000$ to $z + 1$ ſhall be nearer to a proportion of equality, or ſhall be a leſs ratio of majority, than the propoſed ratio of 1 to 0.99999, or of 100,000 to 99,999. And the ſame thing might be done if the propoſed ratio, inſtead of being that of 1 to 0.99999, or of 100,000 to 99,999, had been that of 1 to 0.999,999, or of 1000,000 to 999,999, or that of 1 to 0.999,999,9, or of 10,000,000 to 9,999,999, or any other ratio of majority, how ſmall ſoever. Therefore, however nearly the quantity x may approach to an equality with 1 (than which it is always ſuppoſed to be ſomewhat leſs) it will always be poſſible to increaſe the number z till the proportion of $z + n$ to $z + 1$, or of $n + z$ to $z + 1$, becomes leſs than the proportion of 1 to x, or till the fraction $\frac{n+z}{z+1}$ becomes leſs than the fraction $\frac{1}{x}$. And, as the number z increaſes with the number of terms of the ſeries $1 + \frac{n}{1} Ax + \frac{n+1}{2} Bx^2 + \frac{n+2}{3} Cx^3 + \frac{n+3}{4} Dx^4 + \frac{n+4}{5} Ex^5 + \frac{n+5}{6} Fx^6 + \frac{n+6}{7} Gx^7$ + &c (being always leſs by 2 than the number of the terms from the beginning of the ſeries to the term in which it occurs, including the ſaid term), it is evident, that, by continuing the terms of the ſeries, we muſt always come to a term in which the generating fraction $\frac{n+z}{z+1}$ ſhall be leſs than the fraction $\frac{1}{x}$. And when we are arrived at this term, the next term of the ſeries will be leſs increaſed by being multiplied into the next generating fraction (which will be leſs than the fraction $\frac{1}{x}$) than it will be diminiſhed by being multiplied into the fraction $\frac{x}{1}$, or the reciprocal of the fraction $\frac{1}{x}$; and conſequently it will be leſs than the laſt preceeding term of the ſeries from which it is derived. And therefore, when

when we are come to this term, all the following terms of the ſaid ſeries (which have hitherto been increaſing quantities) will decreaſe continually, and in a greater and greater proportion continually, as the ſeries advances. And conſequently the ſaid ſeries will in all caſes be of a finite magnitude, however nearly the quantity x may approach to an equality with 1. Q. E. D.

End of the Additional Corollaries not contained in the original text of Mr. James Bernoulli, *which began in page* 123.

A SCHOLIUM.

135. We may here take occaſion to obſerve, that, though many writers on mathematical ſubjects (as for example, Faulhaber and Remmelin of the city of Ulm in Germany, and Dr. John Wallis of Oxford, Mr. Nicholas Mercator in his Logarithmotechnia *, and Monſieur Preſtet, a learned French mathematician) have made the properties of the figurate numbers the ſubject of their conſideration, yet no one has hitherto given the publick a general and ſcientifick demonſtration of the foregoing important 12th property of them. At leaſt I may ſay, that no ſuch demonſtration has ever come to my knowledge. Dr. Wallis, indeed, in that part of his learned treatiſe on the arithmetick of infinites, in which he eſtabliſhes the foundations of his method, has inveſtigated by arguments of induction the proportions which a ſeries of the ſquares of a given number of the natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, and a ſeries of their cubes, and a ſeries of their fourth powers, and ſerieſes

* See Vol. I. of the Collection of Tracts, in quarto, intitled, *Scriptores Logarithmici*, pages 192, 193.

of

of their following higher powers, would bear to a ſeries conſiſting of the ſame number of terms all equal to the laſt, or greateſt, term of the former ſeries; and, after performing theſe inveſtigations, has, in his 176th propoſition, made a tranſition to the contemplation of the trigonal, or triangular, and the pyramidal, and trigono-pyramidal, or triangulo-pyramidal, numbers, and other following orders of the figurate numbers. But I apprehend he would have acted more judiciouſly and more agreeably to the nature of the ſubject he was conſidering, if he had taken the contrary courſe, and begun with the inveſtigation of the properties of the figurate numbers, and then, after having diſcovered thoſe properties, and given a juſt and general demonſtration of them, had proceeded to inveſtigate the ſums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. For, beſides the objections that may be juſtly made to his method of making theſe inveſtigations by inductions from particular examples, as being by no means ſcientifick or ſatisfactory to a mind accuſtomed to more accurate modes of reaſoning, and likewiſe as being more prolix and tedious than need be, on account of the neceſſity of having a ſeparate inveſtigation for every new ſeries of powers;—I ſay, beſides theſe objections to his method of treating this ſubject, it may be conſidered as inelegant and unnatural on another account, namely, becauſe it treats of the more abſtruſe parts of the ſubject, to wit, the inveſtigation of the ſums of powers, before the more ſimple and eaſy parts of it, or the doctrine of the figurate numbers. For theſe numbers may be juſtly eſteemed to be more ſimple and eaſy to be underſtood than the powers of the natural numbers, partly, becauſe the ſeveral orders of them are generated one from another by the eaſy operation of addition; whereas, the powers of numbers are produced by the more complicated operation of multiplication; and partly and eſpecially, becauſe the ſums of the ſeveral orders of figurate numbers (reckoning from the beginning of the foregoing table of them, or including the ſeveral cyphers prefixed to the ſignificant terms of the ſeveral vertical columns of the ſaid table) are (as we have ſeen) exact aliquot parts of the ſerieſes that conſiſt of the ſame fame numbers of terms all equal to their laft, or greateft, terms, refpectively; whereas the fums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, never are exact aliquot parts of the fums of equal numbers of terms equal to the laft, or greateft, of them, refpectively, but always exceed, or fall fhort of, fuch aliquot parts by fome fmall finite quantity, how great foever the number of the terms of fuch feriefes may be fuppofed to be, and what number of cyphers foever we may prefix to the faid feriefes confifting of the powers of the natural numbers. Nor can it be alledged, that it was neceffary for Dr. Wallis to begin by inveftigating the fums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, on account of the difficulty of deducing the values of thofe fums from the doctrine of the fums of the figurate numbers. For, it is full as eafy to deduce the fums of the faid powers from the fums of the feveral orders of figurate numbers, as to deduce the latter from the fums of the powers of numbers in the manner adopted by Dr. Wallis: as I fhall now proceed to fhew by deducing the fums of the faid powers from the fums of the feveral orders of figurate numbers, which we have already inveftigated.

An inveftigation of the fum of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, *&c, continued to any given number of terms, and of the fums of their fquares, and of their cubes, and of their fourth powers, and other higher powers, continued to the fame number of terms.*

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136. If the letter x be made to denote the feveral fucceffive terms of the feries 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to the nth term, which, it is evident, will be n, the

the succeffive values of the refidual quantities $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, $x-1$, &c, continued to n terms, will be $1-1$, $2-1$, $3-1$, $4-1$, $5-1$, $6-1$, $7-1$, $8-1$, $9-1$, $10-1$, $11-1$, &c, continued to n terms, or 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c, continued to n terms. But, by coroll. 2, of the foregoing propofition, art. 81, the feries 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c, (which are the terms contained in the fecond vertical column of the foregoing table of combinations), continued to n terms, is $= \frac{n \times \overline{n-1}}{2} = \frac{nn-n}{2} = \frac{nn}{2} - \frac{n}{2}$. Therefore the fum of all the n fucceffive values of $x-1$ will be equal to $\frac{nn}{2} - \frac{n}{2}$; and confequently, if we denote the faid fum by $S.\overline{x-1}$, we fhall have $S.\overline{x-1} = \frac{nn}{2} - \frac{n}{2}$. But the fum of the n fucceffive values of $x-1$ is equal to the excefs of the n fucceffive values of x above the n fucceffive values of 1, or (making ufe of the fame kind of notation) to $S.x - S.1$. Therefore $S.x - S.1$ will be $= \frac{nn}{2} - \frac{n}{2}$, and confequently (adding S.1 to both fides) $S.x$ will be $= \frac{nn}{2} - \frac{n}{2} + S.1$. But the fum of the n fucceffive values of 1 is evidently the number n. Therefore $S.x$ will be $= \frac{nn}{2} - \frac{n}{2} + n = \frac{nn}{2} + \frac{n}{2}$, or the fum of all the n fucceffive values of x, to wit, $1+2+3+4+5+6+7+8+9+10+11+ \&c + n$ will be $= \frac{nn}{2} + \frac{n}{2}$. Q. E. D.

Thus, for example, if n is $= 12$, the fum of the twelve terms of the feries $1+2+3+4+5+6+7+8+9+10+11+12$ will be $= \frac{12 \times 12}{2} + \frac{12}{2} = 12 \times 6 + 6$

 $= 72$

$= 72 + 6 = 78$. And so we shall find it to be by actually adding up the terms.

1
2
3
4
5
6
7
8
9
10
11
12
———
78

Of the sum of the squares of the natural numbers, 1, 2, 3, 4, 5, *&c, continued to any given number* n.

137. Let it now be required to find the sum of the squares of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, continued to n, or the sum of the numbers 1, 4, 9, 16, 25, 36, 49, &c, continued to the nth term, which will be nn. This may be done in the manner following.

Let x be put, as before, for the several successive terms of the series 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, n. Then, since by coroll. 4, of the foregoing proposition, art. 83, the nth term of the third vertical column of the foregoing table of combinations is $= \frac{\overline{n-1} \times \overline{n-2}}{2} = \frac{nn - 3n + 2}{2}$, it follows,

lows, that every xth term of the same vertical column will be $= \frac{xx-3x+2}{2}$, or that, if x be made successively equal to 1, 2, 3, 4, 5, 6, &c, the successive values of the fraction $\frac{xx-3x+2}{2}$, will produce the first, second, third, fourth, fifth, sixth, and other following terms of the same vertical column, which are 0, 0, 1, 3, 6, 10, &c. Thus, for example, if x is $= 1$, we shall have $\frac{xx-3x+2}{2} = \frac{1-3+2}{2} = \frac{0}{2} = 0$; and, if x is $= 2$, we shall have $\frac{xx-3x+2}{2} = \frac{4-6+2}{2} = \frac{0}{2} = 0$; and, if x is $= 3$, we shall have $\frac{xx-3x+2}{2} = \frac{9-9+2}{2} = \frac{2}{2} = 1$; and, if x is $= 4$, we shall have $\frac{xx-3x+2}{2} = \frac{16-12+2}{2} = \frac{6}{2} = 3$; and, if x is $= 5$, we shall have $\frac{xx-3x+2}{2} = \frac{25-15+2}{2} = \frac{12}{2} = 6$; and, if x is $= 6$, we shall have $\frac{xx-3x+2}{2} = \frac{36-18+2}{2} = \frac{20}{2} = 10$; which numbers 0, 0, 1, 3, 6, and 10, are the first six terms of the said third vertical column. And the same thing will be found to be true in any greater number of its terms. But, by the second corollary of the foregoing proposition, art. 81, the sum of all the $n - 2$ significant terms, or, including the two cyphers at the beginning of it, the sum of all the n terms, of the said third vertical column is $= \frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3} = \frac{n \times \overline{nn-3n+2}}{2 \times 3} = \frac{n^3-3nn+2n}{6}$. Therefore the sum of all the n successive values of the fraction $\frac{xx-3x+2}{2}$ will be $= \frac{n^3-3nn+2n}{6}$. But the sum of all the n successive values of $\frac{xx-3x+2}{2}$ is evidently equal to the sum of all the n successive

 values

values of $\frac{xx}{2}$, together with the sum of all the n successive values of $\frac{2}{2}$, or 1, diminished by the sum of all the n successive values of $\frac{3x}{2}$, or (according to our former notation) S. $\overline{\frac{xx-3x+2}{2}}$ is $=$ S. $\frac{xx}{2}$ + S. $\frac{2}{2}$ — S. $\frac{3x}{2}$ = S. $\frac{xx}{2}$ + S. 1 — S. $\frac{3x}{2}$ = S. $\frac{xx}{2}$ + n — S. $\frac{3x}{2}$. Therefore S. $\frac{xx}{2}$ + n — S. $\frac{3x}{2}$ will be $= \frac{n^3-3nn+2n}{6}$. But S. $\frac{3x}{2}$ is evidently $= \frac{3}{2} \times$ S. x. Therefore S. $\frac{xx}{2} + n - \frac{3}{2} \times$ S. x will be $=$ S. $\frac{xx}{2} + n -$ S. $\frac{3x}{2}$, and consequently will be $= \frac{n^3-3nn+2n}{6}$. But it has been shewn in art. 136, that S. x is $= \frac{nn}{2} + \frac{n}{2}$. Therefore $\frac{3}{2} \times$ S. x will be $= \frac{3}{2} \times \overline{\frac{nn}{2} + \frac{n}{2}} = \frac{3nn}{4} + \frac{3n}{4}$. Therefore S. $\frac{xx}{2} + n - \frac{3nn}{4} - \frac{3n}{4}$ will be $= \frac{n^3-3nn+2n}{6}$, or S. $\frac{xx}{2} - \frac{3nn}{4} + \frac{n}{4}$ will be $= \frac{n^3-3nn+2n}{6}$, or S. $\frac{xx}{2} - \frac{9nn}{12} + \frac{3n}{12}$ will be $= \frac{n^3}{6} - \frac{6nn}{12} + \frac{4n}{12}$. Therefore (adding $\frac{9nn}{12}$ to both sides) we shall have S. $\frac{xx}{2} + \frac{3n}{12} = \frac{n^3}{6} + \frac{3nn}{12} + \frac{4n}{12}$; and (subtracting $\frac{3n}{12}$ from both sides) we shall have S. $\frac{xx}{2} = \frac{n^3}{6} + \frac{3nn}{12} + \frac{n}{12} = \frac{n^3}{6} + \frac{nn}{4} + \frac{n}{12}$, and consequently S. xx ($= 2 \times$ S. $\frac{xx}{2}$) $= \frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}$; that is, the sum of all the n squares 1, 4, 9, 16, 25, 36, 49, &c, of the n first natural numbers 1, 2, 3, 4, 5, 6, 7, &c . . . . n will be $= \frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}$, or one-third of the cube of the greatest number n, together with half

half the fquare of the faid number, and a fixth part of the faid number itfelf. Q. E. I.

Thus, for example, if n is $= 12$, we fhall have $nn = 144$, and $n^3 = 1728$, and $\frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}$ $(= \frac{1728}{3} + \frac{144}{2} + \frac{12}{6} = 576 + 72 + 2) = 650$. Therefore the fum of the following twelve numbers, to wit, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, and 144, (which are the fquares of the twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12) will be equal to 650. And fo, upon adding them up together, we fhall find them to be

$$\begin{array}{r} 1 \\ 4 \\ 9 \\ 16 \\ 25 \\ 36 \\ 49 \\ 64 \\ 81 \\ 100 \\ 121 \\ 144 \\ \hline 650 \end{array}$$

Of the sum of the cubes of the natural numbers 1, 2, 3, 4, 5, *&c, continued to any given number* n.

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138. Let x be put, as before, for the several successive terms of the series 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, n. Then, since by coroll. 4, of the foregoing proposition, art. 83, the nth term of the fourth vertical column of the foregoing table of combinations is $= \frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3} = \frac{nn-3n+2}{2} \times \frac{n-3}{3} = \frac{n^3-6nn+11n-6}{6}$; it follows, that every xth term of the same fourth column will be $= \frac{x^3-6xx+11x-6}{6}$, or that, if x be made successively equal to 1, 2, 3, 4, 5, 6, &c, the successive values of the fraction $\frac{x^3-6xx+11x-6}{6}$ will produce the first, second, third, fourth, fifth, sixth, and other following terms of the same vertical column, which are 0, 0, 0, 1, 4, 10, 20, 35, &c. But, by the 2d corollary of the foregoing proposition, art. 81, the sum of all the terms of the said fourth vertical column (the number of which, including the three cyphers at the beginning of it, is n) is $= \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4} = \frac{n^4-6n^3+11nn-6n}{24}$. Therefore the sum of the n successive values of the fraction $\frac{x^3-6xx+11x-6}{6}$ will be $= \frac{n^4-6n^3+11nn-6n}{24}$, or, according to our former notation, $S.\left|\frac{x^3-6xx+11x-6}{6}\right.$ will be $= \frac{n^4-6n^3+11nn-6n}{24}$. But $S.\left|\frac{x^3-6xx+11x-6}{6}\right.$ is $= S.\frac{x^3}{6} - S.\frac{6xx}{6} + S.\frac{11x}{6} - S.\frac{6}{6} = S.$

$\frac{x^3}{6}$

$\frac{x^3}{6} - S.xx + \frac{11}{6} \times S:x - S.1 =$ (by the two foregoing articles 136 and 137) $S.\frac{x^3}{6} - \frac{n^3}{3} - \frac{nn}{2} - \frac{n}{6} + \frac{11}{6} \times \overline{\frac{nn}{2} + \frac{n}{2}} - n = S.\frac{x^3}{6} - \frac{n^3}{3} - \frac{nn}{2} - \frac{n}{6} + \frac{11nn}{12} + \frac{11n}{12} - n = S.\frac{x^3}{6} - \frac{n^3}{3} - \frac{6nn}{12} - \frac{2n}{12} + \frac{11nn}{12} + \frac{11n}{12} - \frac{12n}{12} = S.\frac{x^3}{6} - \frac{n^3}{3} + \frac{5nn}{12} - \frac{3n}{12} = S.\frac{x^3}{6} - \frac{n^3}{3} + \frac{5nn}{12} - \frac{n}{4}$. Therefore $S.\frac{x^3}{6} - \frac{n^3}{3} + \frac{5nn}{12} - \frac{n}{4}$ will be $= \frac{n^4 - 6n^3 + 11nn - 6n}{24}$; and consequently (adding $\frac{n^3}{3} + \frac{n}{4}$ to both sides) $S.\frac{x^3}{6} + \frac{5nn}{12}$ will be $= \frac{n^4 - 6n^3 + 11nn - 6n}{24} + \frac{n^3}{3} + \frac{n}{4} = \frac{n^4 - 6n^3 + 11nn - 6n}{24} + \frac{8n^3}{24} + \frac{6n}{24} = \frac{n^4 + 2n^3 + 11nn}{24}$; and (subtracting $\frac{5nn}{12}$ from both sides) $S.\frac{x^3}{6}$ will be $= \frac{n^4 + 2n^3 + 11nn}{24} - \frac{5nn}{12} = \frac{n^4 + 2n^3 + 11nn}{24} - \frac{10nn}{24} = \frac{n^4 + 2n^3 + nn}{24}$. Therefore $S.x^3$ will be $= 6 \times \frac{n^4 + 2n^3 + nn}{24} = \frac{n^4 + 2n^3 + nn}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$, or the sum of the n successive values of x^3, or of the several cube numbers 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, &c, continued to n^3, will be $= \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$. Q. E. I.

Thus, for example, if n is $= 12$, we shall have $nn = 144$, and $n^3 = 1728$, and $n^4 = 20,736$, and consequently $\frac{n^4}{4}$ ($= \frac{20736}{4}$) $= 5184$, and $\frac{n^3}{2}$ ($= \frac{1728}{2}$) $= 864$, and $\frac{nn}{4}$ ($= \frac{144}{4}$) $= 36$, and $\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$ ($= 5184 + 864 + 36$) $= 6084$. Therefore the sum of the twelve numbers 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, and 1728, (which are the cubes of the first twelve natural numbers 1, 2, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12), will be = 6084. And so we shall find the said sum to be, if we actually add up together the said twelve cube numbers.

$$\begin{array}{r} 1 \\ 8 \\ 27 \\ 64 \\ 125 \\ 216 \\ 343 \\ 512 \\ 729 \\ 1000 \\ 1331 \\ 1728 \\ \hline 6084 \end{array}$$

Of the sum of the fourth powers of the natural numbers 1, 2, 3, 4, 5, *&c, continued to any given number* n.

139. Let x be put, as before, for the several successive terms of the series 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, n.

Then, since by coroll. 4, of the foregoing proposition, art. 83, the nth term of the fifth vertical column of the foregoing table of combinations is $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4}$ $= \frac{n^3 - 6nn + 11n - 6}{6} \times \frac{n-4}{4} = \frac{n^4 - 10n^3 + 35nn - 50n + 24}{24}$, it follows that

that every xth term of the same fifth column will be $= \frac{x^4 - 10x^3 + 35xx - 50x + 24}{24}$, or that, if x be made successively equal to 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, the successive values of the fraction $\frac{x^4 - 10x^3 + 35xx - 50x + 24}{24}$ will produce the first, second, third, fourth, fifth, sixth, and other following terms of the same vertical column, which are 0, 0, 0, 0, 1, 5, 15, 35, 70, &c. But, by the second corollary of the foregoing proposition, art. 81, the sum of all the terms of the said fifth vertical column (the number of which, including the four cyphers at the beginning of it, is n) is $= \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2 \times 3 \times 4 \times 5} = \frac{n^5 - 10n^4 + 35n^3 - 50nn + 24n}{120}$. Therefore the sum of all the n successive values of the fraction $\frac{x^4 - 10x^3 + 35xx - 50x + 24}{24}$ will be $= \frac{n^5 - 10n^4 + 35n^3 - 50nn + 24n}{120}$; or, according to our former notation,

$$S.\overline{\frac{x^4 - 10x^3 + 35xx - 50x + 24}{24}} \text{ will be } = \frac{n^5 - 10n^4 + 35n^3 - 50nn + 24n}{120}.$$

But $S.\overline{\frac{x^4 - 10x^3 + 35xx - 50x + 24}{24}}$ is $= S.\frac{x^4}{24} - S.\frac{10x^3}{24} + S.\frac{35xx}{24} - S.\frac{50x}{24} + S.\frac{24}{24} = S.\frac{x^4}{24} - \frac{10}{24} \times S.x^3 + \frac{35}{24} \times S.xx - \frac{50}{24} \times S\,x + S.1 = S.\frac{x^4}{24} - \frac{10}{24} \times \overline{\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}} + \frac{35}{24} \times \overline{\frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}} - \frac{50}{24} \times \overline{\frac{nn}{2} + \frac{n}{2}} + n = S.\frac{x^4}{24} - \frac{10n^4}{96} - \frac{10n^3}{48} - \frac{10nn}{96} + \frac{35n^3}{72} + \frac{35nn}{48} + \frac{35n}{144} - \frac{50nn}{48} - \frac{50n}{48} + n = S.\frac{x^4}{24} - \frac{5n^4}{48} - \frac{30n^3}{144} - \frac{10nn}{96} + \frac{70n^3}{144} + \frac{70nn}{96} + \frac{35n}{144} - \frac{100nn}{96} - \frac{150n}{144} + \frac{144n}{144} = S.\frac{x^4}{24} - \frac{5n^4}{48} + \frac{40n^3}{144} - \frac{40nn}{96} + \frac{29n}{144}$.

 There-

Therefore $S.\frac{x^4}{24} - \frac{5n^4}{48} + \frac{40n^3}{144} - \frac{40nn}{96} + \frac{29n}{144}$ will be $= \frac{n^5 - 10n^4 + 35n^3 - 50nn + 24n}{120}$; and consequently (adding $\frac{5n^4}{48} + \frac{40nn}{96}$ to both sides) we shall have $S.\frac{x^4}{24} + \frac{40n^3}{144} + \frac{29n}{144}$ $(= \frac{n^5 - 10n^4 + 35n^3 - 50nn + 24n}{120} + \frac{5n^4}{48} + \frac{40nn}{96} = \frac{n^5}{120} - \frac{n^4}{12} + \frac{7n^3}{24} - \frac{5nn}{12} + \frac{n}{5} + \frac{5n^4}{48} + \frac{5nn}{12} = \frac{n^5}{120} - \frac{4n^4}{48} + \frac{7n^3}{24} - \frac{5nn}{12} + \frac{n}{5} + \frac{5n^4}{48} + \frac{5nn}{12}) = \frac{n^5}{120} + \frac{n^4}{48} + \frac{7n^3}{24} + \frac{n}{5}$, and subtracting $\frac{40n^3}{144} + \frac{29n}{144}$ from both sides) $S.\frac{x^4}{24}$ $(= \frac{n^5}{120} + \frac{n^4}{48} + \frac{7n^3}{24} + \frac{n}{5} - \frac{40n^3}{144} - \frac{29n}{144} = \frac{n^5}{120} + \frac{n^4}{48} + \frac{42n^3}{144} + \frac{144n}{720} - \frac{40n^3}{144} - \frac{145n}{720} = \frac{n^5}{120} + \frac{n^4}{48} + \frac{2n^3}{144} - \frac{n}{720}) = \frac{n^5}{120} + \frac{n^4}{48} + \frac{n^3}{72} - \frac{n}{720}$; and consequently $S.x^4$ will be $(= 24 \times \frac{n^5}{120} + 24 \times \frac{n^4}{48} + 24 \times \frac{n^3}{72} - 24 \times \frac{n}{720}) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$; or the sum of the n successive values of x^4, or of the several fourth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n, will be $= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$. Q. E. I.

Thus, for example, if n is $= 12$, we shall have $n^3 = 1728$, and $n^4 = 20,736$, and n^5 $(= n^4 \times n = 20,736 \times 12) = 248,832$, and consequently $\frac{n^5}{5} = \frac{248,832}{5}$, and $\frac{n^4}{2}$ $(= \frac{20,736}{2} = 10,368$, and $\frac{n^3}{3}$ $(= \frac{1728}{3})$ 576, and $\frac{n}{30}$ $(= \frac{12}{30}) = \frac{2}{5}$, and $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ $(= \frac{248,832}{5} + 10,368 + 576 - \frac{2}{5} = \frac{248,830}{5} + 10,368 + 576 = 49,766 + 10,368 + 576) = 60,710$. Therefore the sum of the twelve numbers

bers 1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, 14641, and 20,736, (which are the fourth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12), will be = 60,710. And so we shall find the said sum to be, if we actually add up together the said twelve numbers, or fourth powers of the first twelve natural numbers; which may be done as follows.

$$\begin{array}{r} 1 \\ 16 \\ 81 \\ 256 \\ 625 \\ 1,296 \\ 2,401 \\ 4,096 \\ 6,561 \\ 10,000 \\ 14,641 \\ 20,736 \\ \hline 60,710. \end{array}$$

140. The foregoing examples are, I presume, sufficient to shew how the sums of the several powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to any number n, may be deduced from the sums of the several corresponding orders of figurate numbers contained in the foregoing table of combinations. I shall not therefore add the investigations of the sums of any higher powers of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, than the foregoing ones, but shall only set down the results of the like investigations which I have made for my own satisfaction with respect to the sums of the six next higher powers of those numbers, to wit, the fifth powers, the sixth powers, the seventh powers, the eighth powers, the ninth powers, and the tenth powers of them. These results are as follows.

The ſum of all the fifth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n, is =

$$\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{nn}{12}.$$

The ſum of all the ſixth powers of the ſame numbers is =

$$\frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}.$$

The ſum of all the ſeventh powers of the ſame numbers is =

$$\frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{nn}{12}.$$

The ſum of all the eighth powers of the ſame numbers is =

$$\frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} + \frac{n}{30}.$$

The ſum of all the ninth powers of the ſame numbers is =

$$\frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2} - \frac{3nn}{20}^{*}.$$

And the ſum of all the tenth powers of the ſame numbers is =

$$\frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}.$$

* N. B. In computing the laſt term of this expreſſion (which is equal to the ſum of all the ninth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, continued to n) the author has fallen into a miſtake, having made the ſaid laſt term to be $\frac{nn}{12}$, inſtead of $\frac{3nn}{20}$. I have therefore ſet down $\frac{3nn}{20}$ inſtead of $\frac{nn}{12}$ in this tranſlation. I had not diſcovered this miſtake when the ſheet containing it in the original text of the author, page 32, was printing; or I ſhould have ſet it right before.

Examples of the summation of the fifth, sixth, seventh, eighth, ninth, and tenth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, *&c, continued to* n, *by means of the foregoing expressions.*

Let n be $= 12$.

Then for the sum of the fifth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which are 1; 32; 243; 1,024; 3,125; 7,776; 16,807; 32,768; 59,049; 100,000; 161,051; and 248,832;) we shall have $\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{nn}{12}$, or $\frac{\overline{12}|^6}{6} + \frac{\overline{12}|^5}{2} + \frac{5 \times \overline{12}|^4}{12} - \frac{\overline{12}|^2}{12}$ $(= \frac{12 \times \overline{12}|^5}{6} + \frac{\overline{12}|^5}{2} + 5 \times \overline{12}|^3 - 12 = 2 \times \overline{12}|^5 + \frac{\overline{12}|^5}{2} + 5 \times 1728 - 12 = 2 \times 248,832 + \frac{248,832}{2} + 8640 - 12 = 497,664 + 124,416 + 8628) = 630,708$. And so we shall find the said sum to be, if we add up together the said twelve numbers, or fifth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.

1
32
243
1,024
3,125
7,776
16,807
32,768
59,049
100,000
161,051
248,832
630,708.

And

And for the ſum of the ſixth powers of the firſt twelve natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which powers are 1; 64; 729; 4,096; 15,625; 46,656; 117,649; 262,144; 531,441; 1,000,000; 1,771,561; and 2,985,984), we ſhall have $\frac{n^7}{7}+\frac{n^6}{2}+\frac{n^5}{2}-\frac{n^3}{6}+\frac{n}{42}$, or $\frac{\overline{12}|^7}{7}+\frac{\overline{12}|^6}{2}+\frac{\overline{12}|^5}{2}-\frac{\overline{12}|^3}{6}+\frac{12}{42}$, $=\frac{\overline{12}|^6\times 12}{7}+\frac{\overline{12}|^6}{2}+\frac{\overline{12}|^5}{2}-\frac{\overline{12}|^3}{6}+\frac{2}{7}=\frac{2{,}985{,}984\times 12}{7}+\frac{2{,}985{,}984}{2}+\frac{248{,}832}{2}-\frac{1728}{6}+\frac{2}{7}=\frac{35{,}831{,}808}{7}+1{,}492{,}992+124{,}416-288+\frac{2}{7}=\frac{35{,}831{,}810}{7}+1{,}492{,}992+124{,}416-288=5{,}118{,}830+1{,}492{,}992+124{,}416-288=5{,}118{,}830+1{,}492{,}992+124{,}128)=6{,}735{,}950$. And ſo we ſhall find the ſaid ſum to be, if we add up together the ſaid twelve numbers, or ſixth powers of the firſt twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.

$$
\begin{array}{r}
1\\
64\\
729\\
4{,}096\\
15{,}625\\
46{,}656\\
117{,}649\\
262{,}144\\
531{,}441\\
1{,}000{,}000\\
1{,}771{,}561\\
2{,}985{,}984\\
\hline
6{,}735{,}950.
\end{array}
$$

And

And for the sum of the seventh powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which powers are 1; 128; 2,187; 16,384; 78,125; 279,936; 823,543; 2,097,152; 4,782,969; 10,000,000; 19,487,171; and 35,831,808), we shall have $\frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{nn}{12}$, or $\frac{\overline{12}|^8}{8} + \frac{\overline{12}|^7}{2} + \frac{7 \times \overline{12}|^6}{12} - \frac{7 \times \overline{12}|^4}{24} + \frac{\overline{12}|^2}{12}$ (= $\frac{12 \times \overline{12}|^7}{8} + \frac{\overline{12}|^7}{2} + 7 \times \overline{12}|^5 - \frac{7 \times \overline{12}|^3}{2} + 12 = \frac{3 \times \overline{12}|^7}{2} + \frac{\overline{12}|^7}{2} + 7 \times \overline{12}|^5 - \frac{7 \times \overline{12}|^3}{2} + 12 = \frac{3 \times 35,831,808}{2} + \frac{35,831,808}{2} + 7 \times 248,832 - \frac{7 \times 1728}{2} + 12 = 3 \times 17,915,904 + 17,915,904 + 1,741,824 - 7 \times 864 + 12 = 4 \times 17,915,904 + 1,741,824 - 6048 + 12 = 71,663,616 + 1,741,824 - 6048 + 12 = 73,405,452 - 6048$) = 73,399,404. And so we shall find the said sum to be, if we add up together the said twelve numbers, or seventh powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.

1
128
2,187
16,384
78,125
279,936
823,543
2,097,152
4,782,969
10,000,000
19,487,171
35,831,808
73,399,404.

And

And for the sum of the eighth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12, (which powers are 1; 256; 6,561; 65,536; 390,625; 1,679,616; 5,764,801; 16,777,216; 43,046,721; 100,000,000; 214,358,881; and 429,981,696) we shall have $\frac{n^9}{9}+\frac{n^8}{2}+\frac{2n^7}{3}-\frac{7n^5}{15}+\frac{2n^3}{9}-\frac{n}{30}$, or $\frac{\overline{12}|^9}{9}+\frac{\overline{12}|^8}{2}+\frac{2\times\overline{12}|^7}{3}-\frac{7\times\overline{12}|^5}{15}+\frac{2\times\overline{12}|^3}{9}-\frac{12}{30}$ ($=\frac{12\times\overline{12}|^8}{9}+\frac{\overline{12}|^8}{2}+\frac{2\times\overline{12}|^7}{3}-\frac{7\times\overline{12}|^5}{15}+\frac{2\times\overline{12}|^3}{9}-\frac{2}{5}=\frac{4\times\overline{12}|^8}{3}+\frac{\overline{12}|^8}{2}+\frac{2\times\overline{12}|^7}{3}-\frac{7\times\overline{12}|^5}{15}+\frac{2\times 1728}{9}-\frac{2}{5}=\frac{4\times 429,981,696}{3}+\frac{429,981,696}{2}+\frac{2\times 35,831,808}{3}-\frac{7\times 248,832}{15}+\frac{3456}{9}-\frac{2}{5}=\frac{1,719,926,784}{3}+214,990,848+\frac{71,663,616}{3}-\frac{1,741,824}{15}+384-\frac{2}{5}=573,308,928+214,990,848+23,887,872-\frac{580,608}{5}+384-\frac{2}{5}=812,188,032-\frac{580,610}{5}=812,188,032-116,12$) $=812,071,910$. And so we shall find the said sum to be, if we add up together the said twelve numbers, or eighth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows.

$$\begin{array}{r} 1 \\ 256 \\ 6,561 \\ 65,536 \\ 390,625 \\ 1,679,616 \\ 5,764,801 \\ 16,777,216 \\ 43,046,721 \\ 100,000,000 \\ 214,358,881 \\ 429,981,696 \\ \hline 812,071,910 \end{array}$$

And for the sum of the ninth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12, (which powers are 1; 512; 19,683; 262,144; 1,953,125; 10,077,696; 40,353,607; 134,217,728; 387,420,489; 1,000,000,000; 2,357,947,691; and 5,159,780,352;) we shall have $\frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2} - \frac{3nn}{20}$, or $\frac{\overline{12}|^{10}}{10} + \frac{\overline{12}|^{9}}{2} + \frac{3 \times \overline{12}|^{8}}{4} - \frac{7 \times \overline{12}|^{6}}{10} + \frac{\overline{12}|^{4}}{2} - \frac{3 \times \overline{12}|^{2}}{20}$ $\left(= \frac{12 \times \overline{12}|^{9}}{10} + \frac{\overline{12}|^{9}}{2} + \frac{3 \times \overline{12}|^{8}}{4} - \frac{7 \times \overline{12}|^{6}}{10} + \frac{\overline{12}|^{4}}{2} - \frac{3 \times \overline{12}|^{2}}{20} = \right.$ $\frac{12 \times 5,159,780,352}{10} + \frac{5,159,780,352}{2} + \frac{3 \times 429,981,696}{4} -$ $\frac{7 \times 2,985,984}{10} + \frac{20,736}{2} - \frac{3 \times 144}{20} = \frac{61,917,364,224}{10} + 2,579,890,176 + \frac{1,289,945,088}{4} - \frac{20,901,888}{10} + 10,368 - \frac{3 \times 72}{10} =$ $\frac{61,917,364,224}{10} + 2,579,890,176 + 322,486,272 - \frac{20,901,888}{10} + 10,368 - \frac{216}{10} = \frac{61,917,364,224}{10} - \frac{20,902,104}{10} + 2,579,890,$

176 + 322,486,272 + 10,368 = $\frac{61,896.462,120}{10}$ + 2,902, 386,816 = 6,189,646,212 + 2,902,386,816) = 9,092, 033,028. And so we shall find the said sum to be, if we add up together the said twelve numbers, or ninth powers of the first twelve natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows:

1
512
19,683
262,144
1,953,125
10,077,696
40,353,607
134,217,728
387,420,489
1,000,000,000
2,357,947,691
5,159,780,352
9,092,033,028.

And for the sum of the tenth powers of the first twelve natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12, (which powers are 1; 1,024; 59,049; 1,048,576; 9,765,625; 60,466,176; 282,475, 249; 1,073,741,824; 3,486,784,401; 10,000,000,000; 25,937,424,601; and 61,917,364,224), we shall have $\frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}$, or $\frac{\overline{12}|^{11}}{11} + \frac{\overline{12}|^{10}}{2} + \frac{5 \times \overline{12}|^9}{6} - \overline{12}|^7 + \overline{12}|^5 - \frac{\overline{12}|^3}{2} + \frac{5 \times 12}{66}$ (= $\frac{12 \times \overline{12}|^{10}}{11} + \frac{\overline{12}|^{10}}{2} + \frac{5 \times \overline{12}|^9}{6} - \overline{12}|^7 + \overline{12}|^5 - \frac{\overline{12}|^3}{2} + \frac{10}{11} = \frac{12 \times 61,917,364,224}{11} + \frac{61,917,364,224}{2}$

$\frac{61,917,364,224}{2} + \frac{5 \times 5,159,780,352}{6} - 35,831,808 + 248,832 - \frac{1728}{2} + \frac{10}{11} = \frac{743,008,370,688}{11} + 30,958,682,112 + 5 \times 859,963,392 - 35,831,808 + 248,832 - 864 + \frac{10}{11} = \frac{743,008,370,698}{11} + 30,958,682,112 + 4,299,816,960 - 35,831,808 + 248,832 - 864 = 67,546,215,518 + 30,958,682,112 + 4,299,816,960 - 35,831,808 + 248,832 - 864 = 102,804,963,422 - 35,832,672) = 102,769,130,750$. And so we shall find the said sum to be, if we add up together the said twelve numbers, or tenth powers of the first twelve natural numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. This addition will be as follows:

1
1,024
59,049
1,048,576
9,765,625
60,466,176
282,475,249
1,073,741,824
3,486,784,401
10,000,000,000
25,937,424,601
61,917,364,224
102,769,130,750.

141. If the foregoing expressions of the values of the sums of these several sets of powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n, be set down one under the other in a regular table, the said table will be as follows:

A Table of the values of the fums of the natural numbers 1, 2, 3. 4, 5, 6, 7, 8, 9, 10, 11, *&c, continued to* n *terms, and of the fquares, and the cubes, and the fourth powers, and other following powers of the faid numbers, as far as the tenth powers, inclufively.*

The fum of the firft n terms of the faid feries of natural numbers is equal to

$$\frac{nn}{2} + \frac{n}{2}.$$

The fum of the fquares of the faid n terms is

$$= \frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}.$$

The fum of the cubes of the faid n terms is

$$= \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}.$$

The fum of the fourth powers of the faid n terms is

$$= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} * - \frac{n}{30}.$$

The fum of the fifth powers of the faid n terms is

$$= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} * - \frac{nn}{12}.$$

The fum of the fixth powers of the faid n terms is

$$= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} * - \frac{n^3}{6} * + \frac{n}{42}.$$

The fum of the feventh powers of the faid n terms is

$$= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} * - \frac{7n^4}{24} * + \frac{nn}{12}.$$

The fum of the eighth powers of the faid n terms is

$$= \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} * - \frac{7n^5}{15} * + \frac{2n^3}{9} * - \frac{n}{30}.$$

The fum of the ninth powers of the faid n terms is

$$= \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} * - \frac{7n^6}{10} * + \frac{n^4}{2} * - \frac{3nn}{20}.$$

And the fum of the tenth powers of the faid n terms is

$$= \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} * - n^7 * + n^5 * - \frac{n^3}{2} * + \frac{5n}{66}.$$

The

The law of the generation, or derivation, of the terms of the ſeveral expreſſions ſet down in the foregoing table, one from the other.

---

142. By an attentive conſideration of the foregoing table we may diſcover the law by which the terms of the ſeveral expreſſions of which it conſiſts, may be derived one from the other; after which we ſhall be able to continue the ſaid table to the ſums of the eleventh and twelfth and other higher powers of the numbers 1, 2, 3, 4, 5, 6, &c, without entering into the long trains of reaſoning and making the laborious ſubſtitutions of the ſums already known in the expreſſion of the value of the new ſum, which have been uſed in obtaining the foregoing ſums. This law will be found to be as follows.

Let the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, be ſuppoſed to be continued to any number n; and let it be required to find the ſum of the cth powers of the ſaid n terms, or the value of the ſeries $\overline{1|}^{c} + \overline{2|}^{c} + \overline{3|}^{c} + \overline{4|}^{c} + \overline{5|}^{c} + \overline{6|}^{c} + \overline{7|}^{c} + \overline{8|}^{c} + \overline{9|}^{c} + \overline{10|}^{c} + \overline{11|}^{c} +$ &c, continued to $\overline{n|}^{c}$. Let the capital letters A, B, C, D, &c, be put for the co-efficients of the laſt terms of the ſums of the ſquares, and the fourth powers, and the ſixth powers, and the eighth powers, and the other following even powers of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, already computed, with the ſame ſigns + or − prefixed to them, as are prefixed to the ſaid laſt terms, of which they are the co-efficients. Thus, becauſe the laſt term of the ſum of the ſquares of the numbers 1, 2, 3, 4, 5, 6, 7, &c, is $+ \frac{n}{6}$, or $+ \frac{1}{6} \times n$, A will be $= + \frac{1}{6}$; and, becauſe the

the laſt term of the ſum of the fourth powers of the ſaid numbers is $-\frac{n}{30}$, or $-\frac{1}{30} \times n$, B will be $= -\frac{1}{30}$; and, becauſe the laſt term of the ſum of the ſixth powers of the ſaid numbers is $+\frac{n}{42}$, or $+\frac{1}{42} \times n$, C will be $= +\frac{1}{42}$; and, becauſe the laſt term of the ſum of the eighth powers of the ſaid numbers is $-\frac{n}{30}$, or $-\frac{1}{30} \times n$, D will be $= -\frac{1}{30}$; and, becauſe the laſt term of the ſum of the tenth powers of the ſaid numbers is $+\frac{5n}{66}$, or $+\frac{5}{66} \times n$, E will be $= +\frac{5}{66}$. Theſe being the ſeveral values of the capital letters A, B, C, D, E, &c, the ſeries $\overline{1}|^{c} + \overline{2}|^{c} + \overline{3}|^{c} + \overline{4}|^{c} + \overline{5}|^{c} + \overline{6}|^{c} + \overline{7}|^{c} + \overline{8}|^{c} + \overline{9}|^{c} + \overline{10}|^{c} + \overline{11}|^{c} +$ &c, $+ \overline{n}|^{c}$ will be equal to the ſeries $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times A n^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times B n^{c-3}$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times C n^{c-5}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times D n^{c-7}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6} \times \overline{c-7} \times \overline{c-8}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} \times E n^{c-9}$$

$+$ &c; in which the indexes of the powers of n, after the third term $\frac{c}{2} \times A n^{c-1}$, decreaſe continually by 2, till we come at laſt to n or nn, and the co-efficients of $A n^{c-1}$, $B n^{c-3}$, $C n^{c-5}$, $D n^{c-7}$, $E n^{c-9}$, &c, are formed by the continual

continual multiplication of 1 into the fractions $\frac{c}{2}$, $\frac{c-1}{3} \times \frac{c-2}{4}$, $\frac{c-3}{5} \times \frac{c-4}{6}$, $\frac{c-5}{7} \times \frac{c-6}{8}$, $\frac{c-7}{9} \times \frac{c-8}{10}$, $\frac{c-9}{11} \times \frac{c-10}{12}$, &c, till we come to $\frac{c-\overline{c-1}}{c+1} \times \frac{c-c}{c+2}$, or $\frac{c-c+1}{c+1} \times \frac{c-c}{c+2}$, or $\frac{1}{c+1} \times \frac{c-c}{c+2}$, or $\frac{1}{c+1} \times \frac{0}{c+2}$, which will be = 0. And thus we fhall determine the powers of n in all the terms of the faid feries, and alfo the co-efficients of the faid powers of n in all the terms of the faid feries, except the laft term. And this laft co-efficient may be derived from the co-efficients of the preceeding terms, by an eafy addition or fubtraction, being always the quantity which is neceffary to be added to, or fubtracted from, the refult of all the preceeding co-efficients, in order to make fuch refult become equal to 1. Thus, in the firft fum in the foregoing table, to wit, $\frac{nn}{2}+\frac{n}{2}$, the co-efficient of the laft term $\frac{n}{2}$ is $\frac{1}{2}$, which is the quantity which muft be added to $\frac{1}{2}$, the co-efficient of the firft term $\frac{nn}{2}$, in order to make it equal to 1; and, in the fecond fum, $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$, the co-efficient of the laft term $\frac{n}{6}$ is $\frac{1}{6}$, which is the quantity that muft be added to $\frac{1}{3} + \frac{1}{2}$, or the fum of the co efficients of the firft and fecond terms, $\frac{n^3}{3} + \frac{n^2}{2}$, in order to make it equal to 1; and in the third fum, $\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$, the co-efficient of the laft term is $\frac{1}{4}$, which is the quantity that muft be added to $\frac{1}{4} + \frac{1}{2}$, or the fum of the co-efficients of the firft and fecond terms, $\frac{n^4}{4} + \frac{n^3}{2}$, in order

order to make it equal to 1; and in the fourth sum, $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$, the co-efficient of the last term is $\frac{1}{30}$, which is the quantity which must be subtracted from $\frac{1}{5} + \frac{1}{2} + \frac{1}{3}$, or the sum of the co-efficients of the three first terms $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3}$, in order to make it equal to 1. And the same thing is true with respect to the co-efficients of the last terms of all the following sums *; and consequently the said co-efficients, and the signs + or —, to be prefixed to them, may always be determined, by means of the co-efficients of the preceeding terms of the sums to which they belong. And thus all the terms of the series $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A n^{c-1} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times B n^{c-3} + \frac{c}{2} \times \frac{c-1}{3}$

* And hence it will appear that the co-efficient of the last term of the expression that is equal to the sum of the ninth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n, must be $-\frac{3}{20}$, and not $-\frac{1}{12}$; agreeably to what was observed above in the note at the bottom of page 180. For, as the several terms of that expression preceeding the last term are $\frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2}$, of which the co-efficients are $\frac{1}{10} + \frac{1}{2} + \frac{3}{4} - \frac{7}{10} + \frac{1}{2}$, which are $= \frac{2}{20} + \frac{10}{20} + \frac{15}{20} - \frac{14}{20} + \frac{10}{20}$ or $\frac{22}{20} + \frac{1}{20}$, or $\frac{23}{20}$, or $\frac{20}{20} + \frac{3}{20}$, or $1 + \frac{3}{20}$ (from which it is necessary to subtract the fraction $\frac{3}{20}$ in order to make it become equal to 1), it follows, according to the rule here laid down by the author, that the co-efficient of the last term of the said expression must be $-\frac{3}{20}$, and consequently that the last term of the said expression must be $-\frac{3nn}{20}$. Q. E. D.

X

$\times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6} \times Cn^{c-5} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6} \times \frac{c-5}{7} \times \frac{c-6}{8} \times Dn^{c-7} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6} \times \frac{c-5}{7} \times \frac{c-6}{8} \times \frac{c-7}{9} \times \frac{c-8}{10} \times En^{c-9} + \&c,$ may be determined, and consequently the value of the series $\overline{1}|^{c} + \overline{2}|^{c} + \overline{3}|^{c} + \overline{4}|^{c} + \overline{5}|^{c} + \overline{6}|^{c} + \overline{7}|^{c} + \overline{8}|^{c} + \overline{9}|^{c} + \overline{10}|^{c} + \overline{11}|^{c} + \&c + \overline{n}|^{c}$ (to which the said series is equal) may be assigned, without the help of the long reasonings and laborious substitutions that were employed for this purpose in the foregoing articles. The method of doing this will appear more clearly from the following example.

An example of the computation of the expression that is equal to the sum of certain powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, *&c, continued to* n, *by means of the foregoing series.*

---

143. As an example to the foregoing series, let it be required to find the value of the series $\overline{1}|^{10} + \overline{2}|^{10} + \overline{3}|^{10} + \overline{4}|^{10} + \overline{5}|^{10} + \overline{6}|^{10} + \overline{7}|^{10} + \overline{8}|^{10} + \overline{9}|^{10} + \overline{10}|^{10} + \overline{11}|^{10} + \&c + \overline{n}|^{10}$, or the sum of the 10th powers of the several natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n terms; which sum, as set down in the foregoing table, is $= \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^{9}}{6} - n^{7} + n^{5} - \frac{n^{3}}{2} + \frac{5n}{66}$.

In this case the index c is $= 10$, and consequently $c + 1$ is $= 11$, and $c - 1$ is $= 9$, and $c - 2$, $c - 3$, $c - 4$, $c - 5$,

 $c - 6$,

$c-6$, $c-7$, $c-8$, $c-9$, and $c-10$ are, respectively, equal to 8, 7, 6, 5, 4, 3, 2, 1, and 0. We shall therefore have

$\frac{c}{2} (= \frac{10}{2}) = 5$,

and $\frac{c-1}{3} \times \frac{c-2}{4} (= \frac{9}{3} \times \frac{8}{4} = 3 \times 2) = 6$,

and $\frac{c-3}{5} \times \frac{c-4}{6} (= \frac{7}{5} \times \frac{6}{6}) = \frac{7}{5}$,

and $\frac{c-5}{7} \times \frac{c-6}{8} (= \frac{5}{7} \times \frac{4}{8} = \frac{5}{7} \times \frac{1}{2}) = \frac{5}{14}$,

and $\frac{c-7}{9} \times \frac{c-8}{10} (= \frac{3}{9} \times \frac{2}{10} = \frac{1}{3} \times \frac{1}{5}) = \frac{1}{15}$,

and $\frac{c-9}{11} \times \frac{c-10}{12} (= \frac{1}{11} \times \frac{0}{12}) = 0$.

Therefore the series $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times A n^{c-1} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times B n^{c-3} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6} \times C n^{c-5} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6} \times \frac{c-5}{7} \times \frac{c-6}{8} \times D n^{c-7} + \frac{c}{2} \times \frac{c-1}{3} \times \frac{c-2}{4} \times \frac{c-3}{5} \times \frac{c-4}{6} \times \frac{c-5}{7} \times \frac{c-6}{8} \times \frac{c-7}{9} \times \frac{c-8}{10} \times E n^{c-9}$ will be $= \frac{1}{11} \times n^{11} + \frac{1}{2} \times n^{10} + 5 \times A n^{9} + 5 \times 6 \times B n^{7} + 5 \times 6 \times \frac{7}{5} \times C n^{5} + 5 \times 6 \times \frac{7}{5} \times \frac{5}{14} \times D n^{3} + 5 \times 6 \times \frac{7}{5} \times \frac{5}{14} \times \frac{1}{15} \times E n = \frac{n^{11}}{11} + \frac{n^{10}}{2} + 5 A n^{9} + 30 B n^{7} + 42 C n^{5} + 42 \times \frac{5}{14} \times D n^{3} + 42 \times \frac{5}{14} \times \frac{1}{15} \times E n = \frac{n^{11}}{11} + \frac{n^{10}}{2} + 5 A n^{9} + 30 B n^{7} + 42 C n^{5} + 15 D n^{3} + E n = \frac{n^{11}}{11} + \frac{n^{10}}{2} + 5 \times + \frac{1}{6} \times n^{9} + 30 \times - \frac{1}{30} \times n^{7} + 42 \times + \frac{1}{42} \times n^{5} + 15$

$\times$

$\times - \frac{1}{30} \times n^3 - En = \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2}$ $+ En$; of which expreſſion all the terms, except the laſt term En, are known quantities. And this laſt term En may be found in the following manner. The co-efficients of all the preceeding terms are $\frac{1}{11} + \frac{1}{2} + \frac{5}{6} - 1 + 1 - \frac{1}{2}$, which are equal to $\frac{1}{11} + \frac{5}{6} = \frac{6}{66} + \frac{55}{66} = \frac{61}{66}$; to which it is neceſſary to add $\frac{5}{66}$, in order to make the reſult equal to 1. Therefore E, or the co-efficient of the laſt term En, will be $= + \frac{5}{6}$; and conſequently the compleat value of the foregoing ſeries in this caſe of the 10th powers of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, continued to n terms, will be $\frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66}$; which is the value ſet down for the ſum of the ſaid 10th powers in the foregoing table.

A numerical example of the computation of the sum of the tenth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, *&c, continued to* 1000, *by means of the foregoing expression.*

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144. If n is $=$ 1000, we shall have

n^3 ($= \overline{1000}|^3$)
$=$ 1000,000,000,
and n^5 ($= \overline{1000}|^5$)
$=$ 1000,000,000,000,000;
and n^7 ($= \overline{1000}|^7$)
$=$ 1000,000,000,000,000,000,000,
and n^9 ($= \overline{1000}|^9$)
$=$ 1000,000,000,000,000,000,000,000,000,
and n^{10} ($= \overline{1000}|^{10}$)
$=$ 1000,000,000,000,000,000,000,000,000,000;
and n^{11} ($= \overline{1000}|^{11}$)
$=$ 1000,000,000,000,000,000,000,000,000,000,000,

and consequently

$$\frac{n^{11}}{11} \left(= \frac{1000,000,000,000,000,000,000,000,000,000,000}{11}\right) =$$

90,909,090,909,090,909,090,909,090,909,090,

and $\frac{n^{10}}{2}$ $\left(= \frac{1000,000,000,000,000,000,000,000,000,000}{2}\right) =$

500,000,000,000,000,000,000,000,000,000,

and $\frac{5n^9}{6}$ $\left(= \frac{5}{6} \times 1000,000,000,000,000,000,000,000,000,\right.$

$$\left. = \frac{5000,000,000,000,000,000,000,000,000}{6}\right) =$$

833,333,333,333,333,333,333,333,333·333,333, &c.

and $\frac{n^3}{2}$ $\left(= \frac{1000,000,000}{2}\right) =$ 500,000,000;

and

$$\text{and } \frac{5n}{6}\left(=\frac{5\times 1000}{66}=\frac{5000}{66}\right)=75\cdot757{,}575, \text{ \&c};$$

$$\text{and } \frac{n^{11}}{11}+\frac{n^{10}}{2}+\frac{5n^9}{6}-n^7+n^5-\frac{n^3}{2}+\frac{5n}{66}$$

$$=\left\{\begin{array}{r}
90{,}909{,}090{,}909{,}090{,}909{,}090{,}909{,}090{,}909{,}090\cdot909{,}090, \text{ \&c}\\
+\ 500{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000\cdot000{,}000,\phantom{\text{ \&c}}\\
+\ 833{,}333{,}333{,}333{,}333{,}333{,}333{,}333{,}333\cdot333{,}333, \text{ \&c}\\
-\ 1{,}000{,}000{,}000{,}000{,}000{,}000{,}000\cdot000{,}000,\phantom{\text{ \&c}}\\
+\ 1{,}000{,}000{,}000{,}000{,}000\cdot000{,}000,\phantom{\text{ \&c}}\\
-\ 500{,}000{,}000\cdot000{,}000,\phantom{\text{ \&c}}\\
+\ 75\cdot757{,}575, \text{ \&c}
\end{array}\right.$$

$$=\left\{\begin{array}{r}
91{,}409{,}924{,}242{,}424{,}243{,}424{,}242{,}424{,}242{,}499\cdot999{,}998, \text{ \&c}\\
-\ 1{,}000{,}000{,}000{,}000{,}500{,}000{,}000\cdot000{,}000,\phantom{\text{ \&c}}
\end{array}\right.$$

$$=91{,}409{,}924{,}241{,}424{,}243{,}424{,}241{,}924{,}242{,}499\cdot999{,}998,$$

$$\text{or } 91{,}409{,}924{,}241{,}424{,}243{,}424{,}241{,}924{,}242{,}500.$$

Therefore the fum of all the tenth powers of the firft thoufand natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, ... 1000, is 91,409,924,241,424,243,424,241,924,242,500, or more than 91 quintillions, or 91 times the fifth power of a million.

145. I cannot but obferve on this occafion, that the learned *Ifmael Bullialdus*, or *Bouillaud*, has been rather unfortunate in his manner of treating this fubject, in his Treatife on the Arithmetick of Infinites *; fince the whole of the folio volume which he has written upon it does nothing more than enable us to find the fums of the firft fix powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, continued to any given number n; which is only a part of what we have here accomplifhed in the compafs of a dozen pages.

* See an account of this book of Monfieur Bouillaud in Dr. Wallis's Algebra, chapter lxxx. pages 310, 311.

A com-

A computation of all the other expressions given above in the Table set down in art. 141, *page* 188, *for the values of the sums of the powers of the natural numbers* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, *&c, continued to the number* n, *by means of the foregoing general series* $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} +$

$$\frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{c-7}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6} \times \overline{c-7} \times \overline{c-8}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10} \times En^{c-9}$$

$+$ &c.

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146. If the foregoing series be applied in the same manner to the computation of the sums of the preceeding powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, as it has here been applied to the computation of the sum of their tenth powers, it will be found to produce the several expressions set down above in the table in art. 141, for the values of the sums of those powers; to wit, for the value of the sum of the first, or simple, powers of the said natural numbers, continued to the number n, or for the sum of the said natural numbers themselves, continued to the number n, the expression

$$\frac{nn}{2} + \frac{n}{2};$$

And for the sum of the squares of the said n terms the expression

$$\frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6};$$

And

And for the ſum of the cubes of the ſaid n terms the expreſſion

$$\frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4};$$

And for the ſum of the fourth powers of the ſaid n terms the expreſſion

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} * - \frac{n}{30};$$

And for the ſum of the fifth powers of the ſaid n terms the expreſſion

$$\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} * - \frac{nn}{12};$$

And for the ſum of the ſixth powers of the ſaid n terms the expreſſion

$$\frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} * - \frac{n^3}{6} * + \frac{n}{42};$$

And for the ſum of the ſeventh powers of the ſaid n terms the expreſſion

$$\frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} * - \frac{7n^4}{24} * + \frac{nn}{12};$$

And for the ſum of the eighth powers of the ſaid n terms the expreſſion

$$\frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} * - \frac{7n^5}{15} * + \frac{2n^3}{9} * - \frac{n}{30};$$

And for the ſum of the ninth powers of the ſaid n terms the expreſſion

$$\frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} * - \frac{7n^6}{10} * + \frac{n^4}{2} * - \frac{3nn}{20}.$$

This may be done in the manner following.

In applying this ſeries to the firſt caſe, or the ſum of the firſt, or ſimple, powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to n, or to the ſum of the ſaid natural numbers themſelves, it is evident that we muſt compute only the two firſt terms of the ſaid ſeries, to to wit, the terms $\frac{1}{c+1} \times n^{c+1}$ and $\frac{1}{2} \times n^{c}$; becaufe the following terms involve in them the numbers A, B, C, D, E, &c, which are derived from the values of the fums of the fquares, and the fourth powers, and the fixth powers, and the eighth powers, and the tenth powers, and the other following even powers of the faid natural numbers, with which feveral fums we have as yet nothing to do.

Now, becaufe c is in this cafe $= 1$, the two firft terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c}$ of the foregoing feries will be $= \frac{1}{1+1} \times n^{1+1} + \frac{1}{2} \times n^{1}$ $(= \frac{1}{2} \times n^{2} + \frac{1}{2} \times n) = \frac{nn}{2} + \frac{n}{2}$. Therefore the fum of the firft, or fimple, powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to n, or the fum of the faid natural numbers themfelves, will be $= \frac{nn}{2} + \frac{n}{2}$. Q. E. I.

Secondly, When c is $= 2$, and the fum of the fquares of the faid natural numbers is to be inveftigated by means of the foregoing feries, we muft compute only the three firft terms of the faid feries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times An^{c-1}$; becaufe the following terms involve in them the numbers B, C, D, E, &c, which relate to the fums of the fourth, and the fixth, and the eighth, and the tenth, and the other following even powers of the faid natural numbers, with which fums we have as yet nothing to do.

Now, when c is $= 2$, the three terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times An^{c-1}$ will be $(= \frac{1}{2+1} \times n^{2+1} + \frac{1}{2} \times n^{2} + \frac{2}{2} + An^{2-1} = \frac{1}{3} \times n^{3} + \frac{1}{2} \times n^{2} + 1 \times An^{1})$ =

$= \frac{1}{3} \times n^3 + \frac{1}{2} \times n^2 + A\,n$; of which expreffion the two firft terms are known quantities, and only the third, or laft, term $A\,n$ remains to be inveftigated. Now this laft term $A\,n$ is to be found in the following manner. The co-efficients of the two preceeding terms are $\frac{1}{3} + \frac{1}{2}$, which are equal to $\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$; to which it is neceffary to add $\frac{1}{6}$, in order to make the refult equal to 1. Therefore A, or the co-efficient of the laft term $A\,n$, will be $= + \frac{1}{6}$; and confequently the compleat value of the three firft terms of the foregoing feries in this cafe of the fquares of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, continued to the number n, will be $\frac{1}{3} \times n^3 + \frac{1}{2} \times n^2 + \frac{1}{6} \times n$, or $\frac{n^3}{3} + \frac{nn}{2} + \frac{n}{6}$. Q. E. I.

3dly, When c is $= 3$, and the fum of the cubes of the faid natural numbers is to be inveftigated by means of the foregoing feries, we muft (as in the laft cafe) compute only the three firft terms of the faid feries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A\,n^{c-1}$; becaufe the following terms involve in them the numbers B, C, D, E, &c, which relate to the fums of the fourth, and the fixth, and the eighth, and the tenth, and the other following even powers of the faid natural numbers, with which fums we have as yet nothing to do.

Now, when c is $= 3$, the three terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times A\,n^{c-1}$ will be $= \frac{1}{3+1} \times n^{3+1} + \frac{1}{2} \times n^3 + \frac{3}{2} \times A\,n^{3-1}$ $(= \frac{1}{4} \times n^4 + \frac{1}{2} \times n^3 + \frac{3}{2} \times A n^2$

 =

$= \frac{1}{4} \times n^4 + \frac{1}{2} \times n^3 + \frac{3}{2} \times + \frac{1}{6} \times n^2 = \frac{1}{4} \times n^4 + \frac{1}{2} \times n^3 + \frac{1}{4} \times n^2) = \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$. Therefore the ſum of the cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^4}{4} + \frac{n^3}{2} + \frac{nn}{4}$. Q. E. I.

4thly, When c is $= 4$, and the ſum of the fourth powers of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt only compute the four firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$; becauſe the following terms involve in them the numbers C, D, E, &c, which relate to the ſums of the ſixth, and the eighth, and the tenth, and the other following even powers of the ſaid natural numbers, with which ſums we have hitherto nothing to do.

Now, when c is $= 4$, the four terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$ will be $= \frac{1}{4+1} \times n^{4+1} + \frac{1}{2} \times n^4 + \frac{4}{2} \times An^{4-1} + \frac{4 \times \overline{4-1} \times \overline{4-2}}{2 \times 3 \times 4} \times Bn^{4-3}$ $(= \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + 2 \times An^3 + \frac{4 \times 3 \times 2}{2 \times 3 \times 4} \times Bn^1 = \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + 2 \times An^3 + Bn = \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + 2 \times + \frac{1}{6} \times n^3 + Bn) = \frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + \frac{1}{3} \times n^3 + Bn$; of which expreſſion the laſt term Bn is to be determined in the manner following.

ing. The co-efficients of the three first terms $\frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + \frac{1}{3} \times n^3$ are $\frac{1}{5} + \frac{1}{2} + \frac{1}{3}$ $(= \frac{6}{30} + \frac{15}{30} + \frac{10}{30} = \frac{31}{30}) = 1 + \frac{1}{30}$; from which it is necessary to subtract $\frac{1}{30}$, in order to make the result equal to 1. Therefore B, or the co-efficient of the last term Bn, will be $= -\frac{1}{30}$, and consequently the compleat value of the four first terms of the said series in this case will be $\frac{1}{5} \times n^5 + \frac{1}{2} \times n^4 + \frac{1}{3} \times n^3 - \frac{1}{30} \times n$, or $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$. Therefore the sum of the fourth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} * - \frac{n}{30}$. Q. E. I.

5thly, When c is $= 5$, and the sum of the fifth powers of the said natural numbers is to be investigated by means of the foregoing series, we must, as in the last case, compute only the four first terms of the said series, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$; because the following terms involve in them the numbers C, D, E, &c, which relate to the sums of the sixth, and the eighth, and the tenth, and the other following even powers of the said natural numbers, with which sums we have hitherto nothing to do.

Now, when c is $= 5$, the four terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$ will be $= \frac{1}{5+1} \times n^{5+1} + \frac{1}{2} \times n^5 + \frac{5}{2} \times An^{5-1}$

$+ \frac{5 \times \overline{5-1} \times \overline{5-2}}{2 \times 3 \times 4} \times Bn^{5-3}$ $(= \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{2} \times An^4 + \frac{5 \times 4 \times 3}{2 \times 3 \times 4} \times Bn^2 = \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{2} \times An^4 + \frac{5}{2} \times Bn^2 = \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{2} \times + \frac{1}{6} \times n^4 + \frac{5}{2} \times - \frac{1}{30} \times n^2 = \frac{1}{6} \times n^6 + \frac{1}{2} \times n^5 + \frac{5}{12} \times n^4 - \frac{5}{12} \times n^2) = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{5nn}{12}$. Therefore the sum of the fifth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to n, will be $= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} * - \frac{5nn}{12}$. Q. E. I.

6thly, When c is $= 6$, and the sum of the sixth powers of the said natural numbers is to be investigated by means of the foregoing series, we must compute only the five first terms of the said series, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3} + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$; because the following terms involve in them the numbers D, E, F, G, &c, which relate to the sums of the eighth, and the tenth, and the twelfth, and the fourteenth, and the other following even powers of the said natural numbers, with which sums we have hitherto nothing to do.

Now, when c is $= 6$, the five terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3} + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$ will be $= \frac{1}{6+1} \times n^{6+1}$

$\times n^{6+1} + \frac{1}{2} \times n^6 + \frac{6}{2} \times An^{6-1} + \frac{6 \times \overline{6-1} \times \overline{6-2}}{2 \times 3 \times 4} \times Bn^{6-3} + \frac{6 \times \overline{6-1} \times \overline{6-2} \times \overline{6-3} \times \overline{6-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{6-5}$ $(= \frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + 3 \times An^5 + \frac{6 \times 5 \times 4}{2 \times 3 \times 4} \times Bn^3 + \frac{6 \times 5 \times 4 \times 3 \times 2}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^1 = \frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + 3 \times An^5 + 5 \times Bn^3 + Cn = \frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + 3 \times + \frac{1}{6} \times n^5 + 5 \times -\frac{1}{30} \times n^3 + Cn) = \frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + \frac{1}{2} \times n^5 - \frac{1}{6} \times n^3 + Cn$; of which expression the last term Cn is to be determined in the manner following. The co-efficients of the four first terms of the expression $\frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + \frac{1}{2} \times n^5 - \frac{1}{6} \times n^3 + Cn$ are $\frac{1}{7} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6}$ $(= \frac{6}{42} + \frac{21}{42} + \frac{21}{42} - \frac{7}{42} = \frac{6}{42} + \frac{42}{42} - \frac{7}{42} = \frac{6}{42} + 1 - \frac{7}{42}) = 1 - \frac{1}{42}$; to which it is necessary to add $\frac{1}{42}$, in order to make the result be equal to 1. Therefore C, or the co-efficient of the last term Cn, will be $+\frac{1}{42}$, and consequently the whole expression $\frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + \frac{1}{2} \times n^5 - \frac{1}{6} \times n^3 + Cn$ will be $= \frac{1}{7} \times n^7 + \frac{1}{2} \times n^6 + \frac{1}{2} \times n^5 - \frac{1}{6} \times n^3 + \frac{1}{42} \times n$, or $\frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} * - \frac{n^3}{6} * + \frac{n}{42}$. Therefore the sum of the sixth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} * - \frac{n^3}{6} * + \frac{n}{42}$.

Q. E. I.

7thly,

7thly, When c is $=7$, and the sum of the seventh powers of the said natural numbers is to be investigated by means of the foregoing series, we must, as in the last case, compute only the five first terms of the said series, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3} + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$; because the following terms involve the numbers D, E, F, G, &c, which relate to the sums of the eighth, and the tenth, and the twelfth, and the fourteenth, and the other following even powers of the said natural numbers, with which sums we have hitherto nothing to do.

Now, when c is $=7$, the said five terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^{c} + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3} + \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$ will be $= \frac{1}{7+1} \times n^{7+1} + \frac{1}{2} \times n^{7} + \frac{7}{2} \times An^{7-1} + \frac{7 \times \overline{7-1} \times \overline{7-2}}{2 \times 3 \times 4} \times Bn^{7-3} + \frac{7 \times \overline{7-1} \times \overline{7-2} \times \overline{7-3} \times \overline{7-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{7-5}$ ($= \frac{1}{8} \times n^{8} + \frac{1}{2} \times n^{7} + \frac{7}{2} \times An^{6} + \frac{7 \times 6 \times 5}{2 \times 3 \times 4} \times Bn^{4} + \frac{7 \times 6 \times 5 \times 4 \times 3}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{2}$ $= \frac{1}{8} \times n^{8} + \frac{1}{2} \times n^{7} + \frac{7}{2} \times An^{6} + \frac{7 \times 5}{4} \times Bn^{4} + \frac{7}{2} \times Cn^{2} = \frac{1}{8} \times n^{8} + \frac{1}{2} \times n^{7} + \frac{7}{2} \times + \frac{1}{6} \times n^{6} + \frac{7 \times 5}{4} \times - \frac{1}{30} \times n^{4} + \frac{7}{2} \times + \frac{1}{42} \times n^{2} = \frac{1}{8} \times n^{8} + \frac{1}{2} \times n^{7} + \frac{7}{12} \times n^{6} - \frac{7}{24} \times n^{4} + \frac{1}{12} \times n^{2}$) $= \frac{n^{8}}{8} + \frac{n^{7}}{2} + \frac{7n^{6}}{12} - \frac{7n^{4}}{24} + \frac{nn}{12}$. Therefore the sum of the seventh powers of the

natural

natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} * - \frac{7n^4}{24} * + \frac{nn}{12}$. Q. E. I.

8thly, When c is $= 8$, and the ſum of the eighth powers of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt compute only the ſix firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$ $+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$ $+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{c-7}$; becauſe the following terms involve in them the numbers E, F, G, &c, which relate to the ſums of the tenth, and the twelfth, and the fourteenth, and the other following even powers of the ſaid natural numbers, with which ſums we have hitherto nothing to do.

Now, when c is $= 8$, the ſix terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$ $+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$ $+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{c-7}$ will be $=$ $\frac{1}{8+1} \times n^{8+1} + \frac{1}{2} \times n^8 + \frac{8}{2} \times An^{8-1} + \frac{8 \times \overline{8-1} \times \overline{8-2}}{2 \times 3 \times 4}$ $\times Bn^{8-3} + \frac{8 \times \overline{8-1} \times \overline{8-2} \times \overline{8-3} \times \overline{8-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{8-5}$ $+ \frac{8 \times \overline{8-1} \times \overline{8-2} \times \overline{8-3} \times \overline{8-4} \times \overline{8-5} \times \overline{8-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{8-7}$

$(=\frac{1}{9}\times n^9+\frac{1}{2}\times n^8+4\times An^7+\frac{8\times7\times6}{2\times3\times4}\times Bn^5+\frac{8\times7\times6\times5\times4}{2\times3\times4\times5\times6}\times Cn^3+\frac{8\times7\times6\times5\times4\times3\times2}{2\times3\times4\times5\times6\times7\times8}\times Dn^1=\frac{1}{9}\times n^9+\frac{1}{2}\times n^8+4\times An^7+7\times2\times Bn^5+\frac{7\times4}{3}\times Cn^3+Dn=\frac{1}{9}\times n^9+\frac{1}{2}\times n^8+4\times+\frac{1}{6}\times n^7+7\times2\times-\frac{1}{30}\times n^5+\frac{7\times4}{3}\times+\frac{1}{42}\times n^3+Dn)=\frac{1}{9}\times n^9+\frac{1}{2}\times n^8+\frac{2}{3}\times n^7-\frac{7}{15}\times n^5+\frac{2}{9}\times n^3+Dn$; of which expreſſion the laſt term Dn is to be determined in the manner following. The co-efficients of the five firſt terms of the expreſſion $\frac{1}{9}\times n^9+\frac{1}{2}\times n^8+\frac{2}{3}\times n^7-\frac{7}{15}\times n^5+\frac{2}{9}\times n^3+Dn$ are $\frac{1}{9}+\frac{1}{2}+\frac{2}{3}-\frac{7}{15}+\frac{2}{9}$, which are $(=\frac{10}{90}+\frac{45}{90}+\frac{60}{90}-\frac{42}{90}+\frac{20}{90}=\frac{135}{90}-\frac{42}{90}=\frac{93}{90}=\frac{90}{90}+\frac{3}{90}=1+\frac{3}{90})=1+\frac{1}{30}$; from which it is neceſſary to ſubtract $\frac{1}{30}$, in order to make the reſult be equal to 1. Therefore D, or the co-efficient of the laſt term Dn, will be $=-\frac{1}{30}$, and conſequently the whole expreſſion $\frac{1}{9}\times n^9+\frac{1}{2}\times n^8+\frac{2}{3}\times n^7-\frac{7}{15}\times n^5+\frac{2}{9}\times n^3+Dn$ will be $=\frac{1}{9}\times n^9+\frac{1}{2}\times n^8+\frac{2}{3}\times n^7-\frac{7}{15}\times n^5+\frac{2}{9}\times n^3-\frac{1}{30}\times n=\frac{n^9}{9}+\frac{n^8}{2}+\frac{2n^7}{3}*-\frac{7n^5}{15}*+\frac{2n^3}{9}*-\frac{n}{30}$. Therefore the ſum of the eighth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $=\frac{n^9}{9}+\frac{n^8}{2}+\frac{2n^7}{3}*-\frac{7n^5}{15}*+\frac{2n^3}{9}-\frac{n}{30}$. Q. E. I.

And,

And, 9thly, When c is $= 9$, and the ſum of the ninth powers of the ſaid natural numbers is to be inveſtigated by means of the foregoing ſeries, we muſt, as in the laſt caſe, compute only the ſix firſt terms of the ſaid ſeries, to wit, the terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1}$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5} \times Cn^{c-5}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6}}{2 \times 3 \times 4 \times 5 \times 6} \times Dn^{c-7};$$ becauſe the following terms involve the numbers E, F, G, &c, which relate to the ſums of the tenth, and the twelfth, and the fourteenth, and the other following even powers of the ſaid natural numbers, with which ſums we have hitherto nothing to do.

Now, when c is $= 9$, the ſaid ſix terms $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times Bn^{c-3}$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{c-5}$$

$$+ \frac{c \times \overline{c-1} \times \overline{c-2} \times \overline{c-3} \times \overline{c-4} \times \overline{c-5} \times \overline{c-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{c-7}$$ will be $=$

$$\frac{1}{9+1} \times n^{9+1} + \frac{1}{2} \times n^9 + \frac{9}{2} \times An^{9-1} + \frac{9 \times \overline{9-1} \times \overline{9-2}}{2 \times 3 \times 4} \times Bn^{9-3} + \frac{9 \times \overline{9-1} \times \overline{9-2} \times \overline{9-3} \times \overline{9-4}}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^{9-5}$$

$$+ \frac{9 \times \overline{9-1} \times \overline{9-2} \times \overline{9-3} \times \overline{9-4} \times \overline{9-5} \times \overline{9-6}}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^{9-7}$$

$$(= \frac{1}{10} \times n^{10} + \frac{1}{2} \times n^9 + \frac{9}{2} \times An^8 + \frac{9 \times 8 \times 7}{2 \times 3 \times 4} \times Bn^6 + \frac{9 \times 8 \times 7 \times 6 \times 5}{2 \times 3 \times 4 \times 5 \times 6} \times Cn^4 + \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \times Dn^2 = \frac{1}{10} \times n^{10}$$

$+ \frac{1}{2} \times n^9 + \frac{9}{2} \times An^8 + 7 \times 3 \times Bn^6 + 7 \times 3 \times Cn^4$ $+ \frac{9}{2} \times Dn^2 = \frac{1}{10} \times n^{10} + \frac{1}{2} \times n^9 + \frac{9}{2} \times + \frac{1}{6} \times n^8$ $+ 7 \times 3 \times - \frac{1}{30} \times n^6 + 7 \times 3 \times + \frac{1}{42} \times n^4 + \frac{9}{2} \times$ $- \frac{1}{30} \times n^2$ $(= \frac{1}{10} \times n^{10} + \frac{1}{2} \times n^9 + \frac{3}{4} \times n^8 - \frac{7}{10} \times n^6$ $+ \frac{1}{2} \times n^4 - \frac{3}{20} \times n^2) = \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} * - \frac{7n^6}{10} * +$ $\frac{n^4}{2} * - \frac{3n^2}{20}$. Therefore the sum of the ninth powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, will be $= \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} *$ $- \frac{7n^6}{10} * + \frac{n^4}{2} * - \frac{3nn}{20}$. Q. E. I.

These several expressions of the values of the sums of the first nine powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, continued to the number n, are the same with those set down above in the table in art. 141, page 188. And it is evident that this way of obtaining them, by means of the foregoing general series, is much less laborious than the former method of obtaining them, set forth above in art. 136, 137, 138, and 139.

End of the applications of the foregoing general series $\frac{1}{c+1} \times n^{c+1} + \frac{1}{2} \times n^c + \frac{c}{2} \times An^{c-1} + \frac{c \times \overline{c-1} \times \overline{c-2}}{2 \times 3 \times 4} \times n^{c-3}$ *+ &c, to the investigation of the expressions set down above in art.* 141, *page* 188.

Of

Of certain ſerieſes conſiſting of numbers analogous to the figurate numbers.

147. Before we conclude this chapter, it will not be amiſs to ſhew how certain other ſerieſes, which bear a great reſemblance to the ſerieſes formed by the figurate numbers, may be reduced to, or compared with, the correſponding ſerieſes of thoſe numbers, and how their values, or the ſums of their terms, and likewiſe the values of their laſt terms, may be thereby determined. The ſerieſes I here ſpeak of, and which I call *analogous to the figurate numbers*, are ſuch as have the differences of their terms, or the differences of thoſe differences, or the differences of thoſe ſecond differences, or the differences of the differences of ſome remoter order, equal to each other, and which therefore are generated by the continual addition of a ſet of equal quantities. Let d, d, d, d, d, d, &c, be a ſet of equal quantities, by the continual addition of which to another quantity c we obtain the quantities c, $c + d$, $c + 2d$, $c + 3d$, $c + 4d$, $c + 5d$, &c. And let the terms of this ſecond ſeries c, $c + d$, $c + 2d$, $c + 3d$, $c + 4d$, &c, be continually added to each other, and to a third quantity b, whereby we ſhall obtain a third ſeries of terms, which will be b, $b + c$, $b + 2c + d$, $b + 3c + 3d$, $b + 4c + 6d$, $b + 5c + 10d$, &c; and let the terms of this third ſeries be continually added to each other, and to a fourth quantity a, whereby we ſhall obtain a fourth ſeries of terms, which will be a, $a + b$, $a + 2b + c$, $a + 3b + 3c + d$, $a + 4b + 6c + 4d$, $a + 5b + 10c + 10d$, &c. And let the firſt ſeries d, d, d, d, d, d, &c, be called D; the ſecond ſeries c, $c + d$, $c + 2d$, $c + 3d$, $c + 4d$, $c + 5d$, &c, be called C; the third ſeries b, $b + c$, $b + 2c + d$, $b + 3c + 3d$, $b + 4c + 6d$, $b + 5c + 10d$, &c, be called B; and the fourth, or laſt, ſeries a, $a + b$, $a + 2b + c$, $a + 3b + 3c + d$, $a + 4b + 6c + 4d$, and $a + 5b + 10c + 10d$, &c, be called A. This laſt ſeries A (the firſt differences of the

the terms of which conſtitute the terms of the third ſeries B, and the ſecond differences of the terms of which conſtitute the terms of the ſecond ſeries C, and the third differences of the terms of which conſtitute the terms of the firſt ſeries D, which conſiſts of the equal quantities d, d, d, d, d, d, &c), may, I think, with propriety be called a ſeries *analogous to the figurate numbers*. The generation of the terms of this ſeries will, perhaps, appear more clearly from the following table.

D	C	B	A
d	c	b	a
d	$c+d$	$b+c$	$a+b$
d	$c+2d$	$b+2c+d$	$a+2b+c$
d	$c+3d$	$b+3c+3d$	$a+3b+3c+d$
d	$c+4d$	$b+4c+6d$	$a+4b+6c+4d$
d	$c+5d$	$b+5c+10d$	$a+5b+10c+10d$

148. Now in the laſt ſeries A it is obvious that the co-efficients of the letters a, which are the firſt members of the ſeveral terms a, $a+b$, $a+2b+c$, $a+3b+3c+d$, $a+4b+6c+4d$, and $a+5b+10c+10d$, are a ſet of units, or the firſt order of the figurate numbers; and that the co-efficients of the ſeveral letters b in the ſecond members of the ſaid terms are the lateral, or natural, numbers 1, 2, 3, 4, 5, &c, or the ſecond order of the figurate numbers; and that the co-efficients of the ſeveral letters c in the third members of the ſaid terms are the trigonal, or triangular, numbers 1, 3, 6, 10, &c, or the third order of the figurate numbers; and that the co-efficients of the ſeveral letters d in the fourth members of the ſaid terms are the pyramidal numbers 1, 4, 10, &c, or the fourth order of the figurate numbers. And therefore, as we have above ſhewn how the ſums of the figurate numbers of the ſeveral ſucceſſive orders, and likewiſe the values of the laſt terms in them, may be determined, when the number of terms contained in them is known; it will be eaſy to find both the ſum of all the terms of the ſeries A, by multiplying the ſums of the ſucceſſive columns of figurate numbers, into the letters

a,

a, *b*, *c*, and *d*, respectively, and adding the products so obtained into one sum, and likewise to find the value of the last term of the said series, by multiplying the last terms of the several columns of co-efficients, or figurate numbers, into the letters *a*, *b*, *c*, and *d*, respectively, and adding the said products into one sum. For, if the number of terms in the series A be denoted by the letter *n*, it follows from coroll. 2, art. 81, pages 109, 110, 111, that the sum of the co-efficients of the letter *a* will be *n*; and the sum of the co-efficients of the letter *b* will be $n \times \frac{n-1}{2}$; and the sum of the co-efficients of *c* will be $\frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3}$; and the sum of the co-efficients of *d* will be $\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4}$; and consequently the sum of all the *n* terms of the said series A will be $= n \times a + \frac{n \times \overline{n-1}}{2} \times b + n \times \frac{n \times \overline{n-1} \times \overline{n-2}}{2 \times 3} \times c + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3 \times 4} \times d$. And it follows from coroll. 4, art. 83, pages 112, 113, that the co-efficients of the letters *a*, *b*, *c*, and *d*, in the last, or *n*th, term of the series A will be 1, $n - 1$, $\frac{\overline{n-1} \times \overline{n-2}}{2}$, and $\frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3}$, respectively; and consequently that the said last, or *n*th, term will be $= a + \overline{n-1} \times b + \frac{\overline{n-1} \times \overline{n-2}}{2} \times c + \frac{\overline{n-1} \times \overline{n-2} \times \overline{n-3}}{2 \times 3} \times d$.

Q. E. I *.

* See upon this subject Mr. Thomas Simpson's *Essays on several curious and useful subjects in speculative and mixed mathematics*, published in the year 1740, pages 98, 99, 100, 101, 102, 103, 104, and 105; and likewise his Algebra, 6th Edition, published in the year 1790, Sections XIV and XV, pages 201, 202, &c. — — — 228.

End of the Translation of the foregoing Extract from Mr. James Bernoulli's Treatise De Arte Conjectandi.

A
NEW AND GENERAL METHOD
OF FINDING THE
SUM OF ANY SERIES OF POWERS
OF A SET OF
QUANTITIES THAT ARE IN ARITHMETICAL PROGRESSION;

BEING THE TENTH OF THE LATE LEARNED

MR. THOMAS SIMPSON'S MATHEMATICAL ESSAYS,

PUBLISHED IN THE YEAR 1740.

Mr. SIMPSON's *Tenth Mathematical Essay* *.

PROPOSITION.

To find the sum of any Series of Powers whose roots are in arithmetical progression, as $\overline{m+d}|^n + \overline{m+2d}|^n + \overline{m+3d}|^n \ldots \ldots x^n$, *m, d, and n, being any numbers whatsoever.*

LET $Ax^{n+1} + Bx^n + Cx^{n-1} + Dx^{n-2} + Ex^{n-3} + Fx^{n-4}$, &c. $-$ K, if possible, be always equal to $\overline{m+d}|^n + \overline{m+2d}|^n \ldots\ldots x^n$, and A, B, C, &c, determinate quantities. Then, if any other number in the progression $m+d$, $m+2d$, $m+3d \ldots\ldots x+d$, $x+2d$, $x+3d$, &c, as $x+d$, be substituted instead of x, the equality will still continue; and we shall have

$A \times \overline{x+d}|^{n+1} + B \times \overline{x+d}|^n + C \times \overline{x+d}|^{n-1} + D \times \overline{x+d}|^{n-2}$ &c. $-$ K equal $\overline{m+d}|^n + \overline{m+2d}|^n \ldots \overline{x+d}|^n$; from which if we take the former equation, there will remain $A \times \overline{x+d|^{n+1} - x^{n+1}} + B \times \overline{\overline{x+d}|^n - x^n} + C \times \overline{\overline{x+d}|^{n-1} - x^{n-1}}$, &c. $= \overline{x+d}|^n$, shewing how much each side is increased by augmenting the number of terms in the given series by unity; where, by transposing $\overline{x+d}|^n$, and throwing the several powers of $x+d$ into serieses, we shall have

* This Essay of Mr. Simpson's is the part of his Essays alluded to in the Note at the bottom of page 213. As it is so nearly connected with the subject of the latter part of the foregoing Extract from Mr. James Bernoulli's Treatise *De Arte Conjectandi*, relating to the sums of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, and is not very long, I thought it would be agreeable to the Reader to see it here immediately after the said Extract, and therefore I have caused it to be re-printed. F. M.

$$\left.\begin{array}{l}
-x^{n} \quad -nx^{n-1}d \quad -n\times\frac{n-1}{2}\times x^{n-2}d^{2} \quad -n\times\frac{n-1}{2}\times\frac{n-2}{3}\ x^{n-3}d^{3}, \&c. \\
\overline{n+1}\,Ax^{n}d+\overline{n+1}\times\frac{n}{2}\,Ax^{n-1}d^{2}+\overline{n+1}\times\frac{n}{2}\times\frac{n-1}{3}\,Ax^{n-2}d^{3}+\overline{n+1}\times\frac{n}{2}\times\frac{n-1}{3}\times\frac{n-2}{4}\,Ax^{n-3}d^{4}, \&c. \\
* \quad +nBx^{n-1}d^{2}+ \quad n\times\frac{n-1}{2}\,Bx^{n-2}d^{3}+ \quad n\times\frac{n-1}{2}\times\frac{n-2}{3}\,Bx^{n-3}d^{4}, \&c. \\
* \quad * \quad +\overline{n-1}\times Cx^{n-2}d^{3}+ \quad \overline{n-1}\times\frac{n-2}{2}\,Cx^{n-3}d^{4}, \&c. \\
* \quad * \quad * \quad \overline{n-2}\,Dx^{n-3}d^{4}, \&c. \\
\&c.
\end{array}\right\} = 0.$$

From

From which, by equating the homologous terms, A will come out $= \frac{1}{\overline{n+1} \times d}$, $B = \frac{1}{2}$, $C = \frac{nd}{3.4}$, $D = 0$, $E = -\frac{n \times \overline{n-1} \times \overline{n-2} \times d^3}{2.3.4.5.6}$, $F = 0$, $G = \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times d^5}{2.3.4.5.6.7.6}$ $H = 0$, &c. wherefore the values of A, B, C, &c, being so assigned, the whole expression, or its equal $- \overline{x+d}|^n + A \times \overline{\overline{x+d}|^{n+1} - x^{n+1}} + B \times \overline{\overline{x+d}|^n - x^n}$, &c, must be equal 0, and consequently $A \times \overline{\overline{x+d}|^{n+1} - x^{n+1}} + B \times \overline{\overline{x+d}|^n - x^n}$, &c, $= \overline{x+d}|$; that is, let x and n be what they will, the foresaid increments of $Ax^{n-1} + Bx^n + Cx^{n-1}$, &c, $-$ K and $\overline{m+d}|^n + \overline{m+2d}|^n$, &c. will, under the above assigned values of A, B, &c, be equal to one another: Therefore, if K be taken equal $Am^{n+1} + Bm^n + Cm^{n-1}$, &c, so that when x equal m, or the proposed series is equal to nothing, $Ax^{n+1} + Bx^n$, &c, $-$ K may be also $= 0$, it is manifest, that these two expressions, as they are increased alike, will, in all other circumstances, be equal; that is, let x be what it will, $Ax^{n+1} + Bx^n + Cx^{n-1} + Dx^{n-2}$, &c, $- Am^{n+1} - Bm^n - Cm^{n-1} - Dm^{n-2}$, &c, under the said values of A, B, C, &c, will be always equal to $\overline{m+d}|^n + \overline{m+2d}|^n + \overline{m+3d}|^n \ldots x^n$; which values being therefore substituted, there will be $\frac{x^{n+1}}{\overline{n+1} \times d} + \frac{x^n}{2} + \frac{dnx^{n-1}}{3.4} - \frac{n \times \overline{n-1} \times \overline{n-2}\, d^3 x^{n-3}}{2.3.4.5.6}$ $+ \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{2.3.4.5.6.7.6} d^5 x^{n-5}$

$-\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}}{2.3.4.5.6.7.8.9.10}d^7x^{n-7}$

$+\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}\times\overline{n-7}\times\overline{n-8}}{2.3.4.5.6.7.8.9.11.12}d^9$

x^{n-9}, &c, $-\frac{m^{n+1}}{\overline{n+1}\times d}-\frac{m^n}{2}-\frac{ndm^{n-1}}{3.4}+\frac{n\times\overline{n-1}\times\overline{n-2}}{2.3.4.5.6}\times d^3$

m^{n-3}, &c, $=\overline{m+d}|^n+\overline{m+2d}|^n+\overline{m+3d}|\ldots\ldots x^n$.

Q. E. I.

COROL. I.

Hence, if n be a whole poſitive number, and m be taken equal o; then all the terms in the ſecond ſeries $-\frac{m^{n+1}}{\overline{n+1}\times d}-\frac{m^n}{2}+\frac{ndm^{n-1}}{3.4}$, &c, vaniſhing when n is even, and all but that where the exponent of m is nothing, when odd, we ſhall, in this caſe, have $d^n+\overline{2d}|^n+\overline{3d}|^n+\overline{4d}|^n$ .... x^n barely equal to $\frac{x^{n+1}}{\overline{n+1}\times d}+\frac{x^n}{2}+\frac{ndx^{n-1}}{3.4}-\frac{n\times\overline{n-1}\times\overline{n-2}\,d^3x^{n-3}}{2.3.4.5.6}$, &c, the firſt ſeries continued 'till it terminates, provided that the laſt term, when n is an odd number, be rejected.

COROL. II.

Wherefore, by taking d equal to 1, and n equal to 2, 3, 4, 5, &c, ſucceſſively, we have

1 + 2

$$1 + 2 + 3 + 4 + 5 \ldots + x = \frac{x^2}{2} + \frac{x}{2}$$

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 \ldots + x^2 = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 \ldots + x^3 = \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4}$$

$$1^4 + 2^4 + 3^4 + 4^4 + 5^4 \ldots + x^4 = \frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30}$$

$$1^5 + 2^5 + 3^5 + 4^5 + 5^5 \ldots + x^5 = \frac{x^6}{6} + \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12}$$

$$1^6 + 2^6 + 3^6 + 4^6 + 5^6 \ldots + x^6 = \frac{x^7}{7} + \frac{x^6}{2} + \frac{x^5}{2} - \frac{x^3}{6} + \frac{x}{42}$$

&c. &c.

COROL. III.

Moreover, if d be taken equal to 1, and m equal to 1, our general equation will become $2^n + 3^n + 4^n \ldots + x^n = \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{nx^{n-1}}{3\cdot4}$, &c, $- \frac{1}{n+1} - \frac{1}{2} - \frac{n}{3\cdot4} + \frac{n \times \overline{n-1} \times \overline{n-2}}{2.3.4.5.6}$, &c, each side of which being increased by unity, and the whole multiplied by d^n, gives $d^n + \overline{2d}|^n + \overline{3d}|^n + \overline{4d}|^n \ldots + \overline{dx}|^n = d^n$ into $\frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{nx^{n-1}}{3\cdot4} - \frac{n \times \overline{n-1} \times \overline{n-2}\, x^{n-3}}{2.3.4.5.6}$, &c, $- \frac{1}{n+1} + \frac{1}{2} - \frac{n}{3\cdot4} + \frac{n \times \overline{n-1} \times \overline{n-2}}{2.3.4.5.6}$, &c.

EXAMPLE

EXAMPLE I.

Let it be required to find the sum of a series, consisting of 100 cube numbers, whose roots are, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, &c.

Here d, the common difference of the roots, being equal $\frac{1}{2}$, $n = 3$, and $x = 0$, let these values be substituted in the equation in Cor. II. and it will become ($\overline{\tfrac{1}{2}}|^{3}$ *in*, $\frac{\overline{100}|^{4}}{4} + \frac{\overline{100}|^{3}}{2} + \frac{\overline{100}|^{2}}{4} =$) 3187812.5, the number that was to be found.

EXAMPLE II.

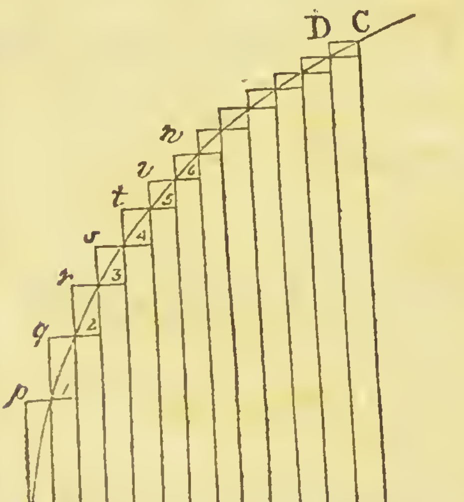
Let $n = \frac{1}{2}$, $d = \frac{1}{4}$. Then the equation in the last Corollary will become $\overline{\tfrac{1}{4}}|^{\frac{1}{2}} + \overline{\tfrac{2}{4}}|^{\frac{1}{2}} + \overline{\tfrac{3}{4}}|^{\frac{1}{2}} \ldots + \overline{\tfrac{x}{4}}|^{\frac{1}{2}} = \overline{\tfrac{1}{4}}|^{\frac{1}{2}} \times \overline{\frac{2 \times x^{\frac{3}{2}}}{3} + \frac{x^{\frac{1}{2}}}{2} + \frac{1}{24 \times x^{\frac{1}{2}}}}$, &c, $- \frac{339}{1920}$ very nearly; so that, taking x equal 4, it will be $\overline{\tfrac{1}{4}}|^{\frac{1}{2}} + \overline{\tfrac{2}{4}}|^{\frac{1}{2}} + \overline{\tfrac{3}{4}}|^{\frac{1}{2}} + 1 = 3.0731$; which differs from the true value by less than $\frac{1}{10000}$; and if more terms had been used, the answer would still have been more exact; but never can come *accurately* true, when n is negative or a fraction, because then both serieses run on *ad infinitum*.

SCHOLIUM.

SCHOLIUM.

The theorems, above found, are not only useful in finding the sum of a Series of Powers, but may be of service also in the Quadrature of Curves, &c, especially as the conclusions will be accurately true, and the reasoning thereupon scientific.

This I shall endeavour to shew by the following instance; wherein A C, being supposed a curve, whose equation is $y = z^n$ (A B being equal z, and C B equal y) the area A B C is required.



Let A B be divided into any number, x, of equal parts, as Ab, bc, cd, &c, and from the points of division let perpendiculars be raised, cutting the curve in the points, 1, 2, 3, &c, and having made p 1, q 2, r 3, s 4, &c, parallel to A B, let the base Ab, bc, cd, &c, of each of the rectangles pb, qc, rd, &c, be represented by d: Then b 1, c 2, d 3, &c, the heights of those rectangles, being ordinates to the curve, will be d^n, $\overline{2d}|^n$, $\overline{3d}|^n$, &c, respectively, each of which ∵ being multiplied by d, the common base, and the sum of all the products taken, will give d into $d^n + \overline{2d}|^n + \overline{3d}|^n \ldots \overline{xd}|^n$, (= A$p$ 1 q 2 r, &c, C B A) for the area of the whole circumscribing polygon; and this series, according to the above said Theorem (Cor. III.) is equal to d^{n+1} *in*, $\frac{nx^{n+1}}{n+1} + \frac{x^n}{2}$, &c, $= \frac{\overline{dx}|^{n+1}}{n+1}$

+

$+ \frac{d \times \overline{dx}|^n}{2}$, &c, or, becauſe $dx = z$, it will be $= \frac{z^{n+1}}{n+1} + \frac{dz^n}{2}$, &c. Now, if from this the difference of the inſcribed and circumſcribed polygons, or the rectangle BD $= dz^n$ be taken, there will remain $\frac{z^{n+1}}{n+1} - \frac{dz^n}{2}$, for the area of the inſcribed polygon. Hence, it is manifeſt, that, let d be what it will, the inſcribed polygon can never be ſo great, nor the circumſcribed ſo ſmall, as $\frac{z^{n+1}}{n+1}$ $(= \frac{AB \times BC}{n+1})$: And therefore this expreſſion muſt be *accurately* equal to the required curvilinear area ACB.

A CON-

AN

INVESTIGATION

AND

DEMONSTRATION

OF

SIR ISAAC NEWTON'S BINOMIAL THEOREM,

IN THE CASE OF

INTEGRAL AND AFFIRMATIVE POWERS;

IN WHICH

The Law of the generation of the numeral co-efficients of the Series which is equal to the quantity $\overline{a+b}|^m$, is difcovered by a conjecture grounded on the obfervation of fome particular inftances; but, when fo difcovered, is fhewn to be true univerfally in all other Integral and Affirmative Powers whatfoever, by a ftrict and accurate Demonftration.

 A General

A General Statement, or Expression, of the Binomial Theorem.

Art. 1. SIR ISAAC NEWTON's Binomial Theorem is a Propofition affirming that, if m be any number whatfoever, either integral or fractional, affirmative or negative, the quantity $\overline{a+b}|^{m}$, or the mth power of the binomial quantity $a+b$, will be equal to the feries $a^{m} + \frac{m}{1} a^{m-1}b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2}b^{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m-3}{4} a^{m-4}b^{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5}b^{5}$ + &c, or (if we put A for 1, or the co-efficient of the firft term a^{m}, and B for $\frac{m}{1}$, or the co-efficient of the fecond term $\frac{m}{1} \times a^{m-1}b$, and C for $\frac{m}{1} \times \frac{m-1}{2}$, or the co-efficient of the third term $\frac{m}{1} \times \frac{m-1}{2} \times a^{m-2}b^{2}$, and D, E, F, G, H, I, K, &c, for the co-efficients of the fourth, fifth, fixth, feventh, eighth, ninth, tenth, and other following terms, refpectively), to the feries $a^{m} + \frac{m}{1} A a^{m-1}b + \frac{m-1}{2} B a^{m-2}b^{2} + \frac{m-2}{3} C a^{m-3}b^{3} + \frac{m-3}{4} D a^{m-4}b^{4} + \frac{m-4}{5} E a^{m-5}b^{5} + \frac{m-5}{6} F a^{m-6}b^{6}$

$+ \frac{m-6}{7} Ga^{m-7}b^7 + \frac{m-7}{8} Ha^{m-8}b^8 + \frac{m-8}{9} I a^{m-9} b^9 +$ &c, in which ſeries the powers of a, to wit, a^m, a^{m-1}, a^{m-2}, a^{m-3}, a^{m-4}, a^{m-5}, a^{m-6}, a^{m-7}, a^{m-8}, a^{m-9}, &c, are produced from each other by a continual diviſion by a, and the powers of b, to wit, b, b^2, b^3, b^4, b^5, b^6, b^7, b^8, b^9, &c, are produced from each other by a continual multiplication by b, and the numeral co-efficients B, C, D, E, F, G, H, I, K, &c, of the ſecond, third, fourth, fifth, ſixth, ſeventh, eighth, ninth, tenth, and other following terms are derived, or generated, from 1, or A, the numeral co-efficient of the firſt term a^m, by a continual multiplication of it into the fractions $\frac{m}{1}$, or $\frac{m-0}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, $\frac{m-8}{9}$, &c, continued *ad infinitum*, or to the end of the ſeries when the number of its terms is finite.

Of the Invention of the ſaid Theorem.

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2. Sir Iſaac Newton was the firſt perſon that expreſſed this important Theorem in the foregoing ſhort and convenient Algebräick notation, and likewiſe the firſt perſon that diſcovered that it would be true, not only when the index m of the power to which the binomial quantity is to be raiſed is a whole number, as 2, 3, 4, 5, 6, &c, but alſo when it is a fraction of any kind, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{17}$, or $\frac{2}{3}$, $\frac{3}{5}$, $\frac{4}{17}$, or

$\frac{3}{2}$,

$\frac{3}{2}$, $\frac{5}{3}$, $\frac{17}{4}$, &c, or even a negative quantity, as -2, -3, -5, -17, or $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{17}$, or $-\frac{2}{3}$, $-\frac{3}{5}$, $-\frac{4}{17}$, or $-\frac{3}{2}$, $-\frac{5}{3}$, $-\frac{17}{4}$, &c.. But he was not the firſt perſon that diſcovered it to be true in the firſt, or ſimpleſt, caſe, or when the index m is equal to an integral and affirmative number. For in that caſe it was known to Mr. Henry Briggs, the celebrated improver and computer of Logarithms, above 40 years before it was diſcovered by Sir Iſaac Newton; and it was publiſhed by Mr. Briggs, in his learned Treatiſe on Logarithms, intitled, *Arithmetica Logarithmica*, in the year 1624; as has been clearly ſhewn by the learned Dr. Hutton, of Woolwich Academy, in his very curious, hiſtorical, Introduction to the new edition of Sherwin's Mathematical Tables, publiſhed in the year 1784.

3. But, though Mr. Briggs had publiſhed this famous Theorem, in this firſt caſe of it, in his *Arithmetica Logarithmica*, in the year 1624, yet it ſeems to have been but little known to Mathematicians till about 60 years after. For even the famous Dr. John Wallis, of Oxford, (who was a very extenſive reader of Mathematical Works, as well as a great improver of the Science,) appears to have been ignorant of it till a little before the year 1685, in which he publiſhed his learned, hiſtorical, Treatiſe of Algebra, at which time he was about 69 years old. For he there tells us, in page 319, that he had formerly ſought to diſcover the law by which the numeral co-efficients of the terms of the ſeries which is equal to $\overline{a+b}|^m$ are generated from each other, but had not been able to find it; and that he had lately been made acquainted with it by the peruſal of a very learned letter of Mr. Iſaac Newton, the Profeſſor of Mathematicks in the Univerſity of Cambridge, to Mr. Oldenburgh, Secretary to the Royal Society, written in the year 1676. His words are theſe, after ſpeaking of ſome other excellent inventions in the Mathematicks contained in the ſaid letter—

"*He*

" *He* [Mr. Newton] *then observes (what I had formerly* " *sought after, but unsuccessfully), that the following numbers* " *are, from the two first, to be found by continual multiplication* " *of this series* $1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$ " *&c.*" From this passage of Dr. Wallis's Algebra, I am inclined to think that this famous theorem was never generally known to Mathematicians till this publication of it in that work. And from its having thus been communicated to the learned world as a discovery of Mr. Newton (who was afterwards better known by the title of Sir Isaac Newton), it has usually been called *his* Theorem.

4. This Theorem had been discovered by Sir Isaac Newton about the year 1665, as appears from his letters to Mr. Oldenburgh in the year 1676, copies of which were sent to Mr. Leibnitz by Mr. Newton's direction. But these letters do not appear to have been known to the Mathematical world in general, till the year 1712, when they were printed in the *Commercium Epistolicum* by the order of the Royal Society. And no part of them seems to have been published before the year 1685, when the foregoing account of the generation of the numeral co-efficients of the terms of the series that is equal to the mth power of a binomial quantity, and a few more curious discoveries contained in them, were inserted by Dr. Wallis, in his Treatise on Algebra.

5. It has been observed above, that Mr. Briggs, and not Sir Isaac Newton, was *the first inventor* of this Theorem in the first and simplest case of it, or when the index m is an affirmative whole number. Yet I am inclined to think that Sir Isaac Newton was likewise *an inventor* of it even in that case, though *not the first inventor*. For it is well known that he was not an extensive reader of Mathematical Works; and he appears to have applied himself principally in his younger years to the study of Des Cartes's Geometry, with Schooten's Commentary on it, and the other Tracts published by Schooten with it, and of Dr. Wallis's *Arithmetica Infinitorum,* and his other works on mathematical subjects then

then publiſhed; in none of which books is any mention made of this uſeful Theorem that had been diſcovered ſo many years before by Mr. Briggs. And, as theſe were the books to which Mr. Newton is known to have given the greateſt part of his attention at that time, he may very well be ſuppoſed not to have ſeen Mr. Briggs's *Arithmetica Logarithmica*, in which this Theorem is contained, at the time of his diſcovering it himſelf, which was about the year 1665, or when he was only 23 years old. And, if he had ſeen that book, and obſerved this diſcovery to be contained in it, I can hardly conceive that, when he was ſpeaking of this Theorem, and ſetting forth its great utility in mathematical inveſtigations, he would have omitted to make mention of the name of Mr. Briggs, and to acknowledge that what he had delivered upon the ſubject in his *Arithmetica Logarithmica* contained the ſubſtance of the ſaid Binomial Theorem in the caſe of Integral Powers, though not expreſſed in Algebräick Symbols. For theſe reaſons I am inclined to think that Sir Iſaac Newton had not ſeen Mr. Briggs's *Arithmetica Logarithmica* when he invented the Binomial Theorem, and conſequently that he was truly *an inventor* of it even in the caſe of Integral Powers, though *not the firſt inventor.*

6. But it ſeems more ſurpriſing that Dr. Wallis, who was a much more copious reader of Mathematical Works than Sir Iſaac Newton, and who actually had ſeen and read Mr. Briggs's *Arithmetica Logarithmica*, and makes mention of it in his Algebra, chapter XII, page 60, ſhould not have attended to the contents of that ingenious Treatiſe enough to have obſerved that it contained this moſt uſeful Theorem. Yet this appears to have been the fact, from what the Doctor tells us in the 85th chapter of his Algebra, page 319, in the paſſage that has been already cited in art. 3, where he mentions the law of the generation of the co-efficients of the terms of the ſeries that is equal to the *m*th power of a binomial quantity, as a diſcovery that had recently come to his knowledge by the peruſal of Mr. Newton's letter to Mr. Oldenburgh. For, ſurely, it muſt be concluded from this declaration, that, though he had ſeen Mr. Briggs's *Arithmetica*

metica Logarithmica, he had not read it with sufficient attention to discover that this method of generating the co-efficients of the terms of the series that is equal to the *m*th power of a binomial quantity, when *m* was equal to any whole number whatsoever, was contained in it: though it seems indeed unaccountably strange that he should not have taken notice of it.

7. We may therefore, upon the whole matter, consider the Binomial Theorem, in the case of Integral and Affirmative Powers, as having been first invented by Mr. Briggs before the year 1624, and published by him in that year in his *Arithmetica Logarithmica,* but in such a manner, and in such expressions, as did not much engage the attention of Mathematicians towards it; since it does not seem to have been generally known amongst Mathematicians till it was afterwards published in the year 1685, as an invention of Mr. Isaac Newton, by Dr. Wallis in his Algebra. And we may consider it as having been invented a second time by Mr. Newton about the year 1665, and extended by him at the same time to the other cases of Fractional and Negative Powers, and also expressed in the very short and convenient Algebräick notation, in which it is set forth above in art. 1, and which has contributed so much to give it currency amongst Mathematicians. And, lastly, we may consider it as having been communicated by Mr. Newton to Mr. Oldenburgh and Mr Leibnitz, and probably also to his friend and patron Dr. Isaac Barrow, the Master of Trinity College, Cambridge, and a few more of his Mathematical friends, in the year 1676, in the letter above-mentioned; and as having afterwards been communicated to the world at large in the aforesaid extracts from the said letter to Mr. Oldenburgh, which Dr. Wallis published in his Algebra in the year 1685.

Of

Of Mr. James Bernoulli's demonstration of the said Theorem, in his Treatise on the Doctrine of Chances, intitled, De Arte Conjectandi.

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8. But, by what steps, or what train of reasoning, Sir Isaac Newton discovered this law of the said co-efficients to be such as he described it, is not known; nor is any demonstration of it, even in the easiest case of it (or when the index m of the power to which the binomial quantity is to be raised, is a whole number), any where to be found in all his works. Nor has Dr. Wallis attempted to supply this defect, nor, as I believe, any other mathematical author whatsoever in the last century, from the year 1685 (when the Theorem was first published by Dr. Wallis) to the end of it; nor do I know of any demonstration of it given in the beginning of the present century before the year 1713, when the learned and sagacious Mr. James Bernoulli's excellent Treatise on the Doctrine of Chances, intitled, *De Arte Conjectandi*, was published at Basil, or Basle, in Switzerland. But there we find an excellent demonstration of it, in the case of Integral Powers, derived from the doctrine of Permutations and Combinations, and the properties of the Figurate numbers, which are the true principles to which it ought to be referred. This demonstration is contained in the 3d chapter of the second part of that valuable Treatise, and may be perfectly understood by a careful perusal of the three first chapters of that second part, without the help of the first part of the Treatise. For the doctrine of Permutations and Combinations is explained from its first principles in the two first chapters of that second part of the Treatise, without any reference to the first part; and the properties of the Figurate numbers are derived from that doctrine in a most strict and scientific manner, in the third chapter of the same second part; and amongst these proper-

 ties

ties of the Figurate numbers, ſo derived, is the Binomial Theorem, in the caſe of Integral and Affirmative Powers, or the law of the generation of the numeral co-efficients of the terms of a ſeries that is equal to any integral and affirmative power of the binomial quantity $a + b$. This demonſtration therefore deſerves to be generally known and ſtudied by Mathematicians, as the true foundation of this celebrated and moſt uſeful Theorem. And upon that account I have re-printed the ſaid three firſt chapters of the ſecond Part of that excellent Treatiſe *De Arte Conjectandi*, in the foregoing part of this volume, in the author's original Latin text, with ſome explanatory notes on a few of the moſt difficult paſſages of it, and have afterwards added a very full Tranſlation of the ſame three chapters, with ſome examples and additions of my own, which I thought might be uſeful to my readers, and which I have taken care to diſtinguiſh from the other parts which are tranſlated from the Author's text. And I hope that, by thus exhibiting this part of that excellent work in an Engliſh dreſs, and removing the difficulties that occur in the original, in conſequence of the Author's extreme conciſeneſs, I ſhall induce the young Students of the Mathematicks in England, to make themſelves acquainted with this maſterly and ſcientific demonſtration of this moſt important Theorem, which ſeems hitherto to have been adopted by too many Mathematicians, upon the mere ground of induction, and the experience of its truth in the ſeveral trials they have made of it, without endeavouring to find a demonſtration of it.

Another

Another demonſtration of it, in the ſame caſe of Integral Powers, will be given in the enſuing part of this Diſcourſe.

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9. But, though the demonſtration of this propoſition given by Mr. James Bernoulli in this excellent Treatiſe, *De Arte Conjectandi*, Part 2d, Chapter 3d, (and which may be ſeen above in this volume in the original Latin text of Mr. James Bernoulli in page 28, and in my Tranſlation of it in pages 115 and 116), is the firſt, and, in my opinion, the beſt that has yet been given of it, yet I doubt not that the Mathematical Reader will be pleaſed to ſee another demonſtration of it, that is ſomewhat ſhorter than Mr. Bernoulli's (inaſmuch as it does not require the previous knowledge of the Doctrine of Permutations and Combinations, and the properties of the Figurate Numbers), and yet is equally accurate and concluſive. Such a demonſtration I ſhall therefore now endeavour to lay before him in the remaining part of this Diſcourſe.

10. Now in order to diſcover the general relation of the terms of the ſeries that is equal to $\overline{a+b}|^m$ to each other, when m denotes any whole number whatſoever, it will be proper in the firſt place to examine their relation to each other when m is equal to ſome particular whole numbers, and thoſe not large ones, that they may be more eaſily managed and their properties more readily ſeen into. And, if, when we have examined theſe particular ſerieſes that are equal to certain particular values of $\overline{a+b}|^m$, when m is equal to certain ſmall whole numbers, we can find any common properties that belong equally to all of them, and can alſo perceive that the ſame properties muſt likewiſe belong to all the ſerieſes that ſhall be equal to any other values of $\overline{a+b}|^m$, as well as to thoſe which we have conſidered; or, if we cannot immediately perceive this to be the caſe, but can find ſome method of demonſtrating that it is ſo; we ſhall then arrive at the

 knowledge

knowledge of the general relation of the terms of the series that is equal to $\overline{a+b}|^m$, to each other, which is the object of our pursuit. We will therefore raise the binomial quantity $a+b$ to its square, and cube, and fourth power, and fifth power, and sixth power, by multiplication; which may be done in the manner following.

$$1 = \overline{a+b}|^0.$$

$$\begin{array}{l}
a+b \\
\hline
a+b = \overline{a+b}|^1. \\
a+b \\
\hline
aa+ab \\
\quad +ab+bb \\
\hline
aa+2ab+bb = \overline{a+b}|^2. \\
\qquad a+b \\
\hline
a^3+2a^2b+ab^2 \\
\quad +a^2b+2ab^2+b^3 \\
\hline
a^3+3a^2b+3ab^2+b^3 = \overline{a+b}|^3. \\
\qquad\qquad a+b \\
\hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\quad +a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
a^4+4a^3b+6a^2b^2+4ab^3+b^4 = \overline{a+b}|^4. \\
\qquad\qquad\qquad a+b \\
\hline
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\quad +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\
\hline
a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 = \overline{a+b}|^5. \\
\qquad\qquad\qquad\qquad a+b \\
\hline
a^6+5a^5b+10a^4b^2+10a^3b^3+5a^2b^4+ab^5 \\
\quad +a^5b+5a^4b^2+10a^3b^3+10a^2b^4+5ab^5+b^6 \\
\hline
a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6 = \overline{a+b}|^6.
\end{array}$$

Observa-

Observations on the terms of the foregoing Serieses that are equal to $\overline{a+b}|^1$, $\overline{a+b}|^2$, $\overline{a+b}|^3$, $\overline{a+b}|^4$, $\overline{a+b}|^5$, and $\overline{a+b}|^6$, explaining the Composition of the Literal parts of the said terms.

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11. If we examine the composition of the foregoing products, or serieses, which are obtained by continual multiplications by the binomial quantity $a+b$, the first observation that will occur to us will be, that the first term of the series $aa+2ab+bb$, which is equal to the square of $a+b$, is aa or a^2; and that the first term of the series $a^3+3a^2b+3ab^2+b^3$, which is equal to the cube of $a+b$, is a^3; and that the first term of the series $a^4+4a^3b+6a^2b^2+4ab^3+b^4$, which is equal to the fourth power of $a+b$, is a^4; and that the first term of the series $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, which is equal to the fifth power of $a+b$, is a^5; and that the first term of the series $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$, is a^6; or that the first term of the series that is equal to any one of the said five powers of the binomial quantity $a+b$, is the same power of the single quantity a.

And it is easy to see that, if we were to continue these multiplications by the binomial quantity $a+b$ ever so far, the same thing would take place with respect to the first terms of the following products, or serieses, which are equal to any higher powers of the binomial quantity $a+b$, let their number be ever so great; or that, if the letter m be any number, how great soever, the first term of the product, or series, that is equal to $\overline{a+b}|^m$, or the mth power of the binomial quantity $a+b$, will be a^m, or the same power of the single quantity a.

For, as the first term of every new product, or series, is produced by the multiplication of the first term of the next preceeding

preceeding product, or series, by a, or $1 \times a$, the co-efficient of the first term of the new series, which is the product of the said multiplication, must be the same with the co-efficient of the first term of the next preceeding series, which is the multiplicand of the said multiplication. And consequently, as the co-efficient of the said multiplicand, or first term of the preceeding series is originally 1, (namely, when $a+b$ is multiplied into $a+b$, in order to produce the series $aa+2ab+bb$, which is equal to its square), the co-efficients of the first terms of all the following products, or serieses, which are equal to $\overline{a+b}|^2$, $\overline{a+b}|^3$, $\overline{a+b}|^4$, $\overline{a+b}|^5$, $\overline{a+b}|^6$, $\overline{a+b}|^7$, $\overline{a+b}|^8$, $\overline{a+b}|^9$, $\overline{a+b}|^{10}$, &c, *ad infinitum*, must likewise all be equal to 1. Q. E. D.

12. The second observation that will occur to us, is, that the indexes of the several successive powers of a in the terms of every product, or series, that is equal to any power of the binomial quantity $a+b$, decrease continually by an unit, and that the indexes of the powers of b in the terms of the said products, or serieses, increase by an unit at the same time. Thus, in the series $a^2+2ab+b^2$, or (as it is sometimes expressed in Sir Isaac Newton's Notation of Indexes, because a^0 is $=1$, and b^0 is likewise $=1$), $a^2b^0+2a^1b^1+a^0b^2$, which is equal to the square of the binomial quantity $a+b$, the indexes of the powers of a are 2, 1, and 0, or 2, $2-1$, and $2-2$, and the indexes of the powers of b are 0, 1, and 2, or 0, $0+1$, and $0+2$; and in the series $a^3+3a^2b+3ab^2+b^3$, or $a^3b^0+3a^2b^1+3a^1b^2+a^0b^3$, which is equal to the cube of the binomial quantity $a+b$, the indexes of the powers of a are 3, 2, 1, and 0, or 3, $3-1$, $3-2$, and $3-3$, and the indexes of the powers of b are 0, 1, 2, and 3. And the same thing takes place in the following products, or serieses, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$, and $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, and $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$, which are equal to the fourth, fifth, and sixth powers of $a+b$. And it is easy to see that the same thing will likewise take place in the terms of the products, or serieses, that are equal

to

to any higher powers of $a+b$ whatſoever, if the ſaid multiplications by $a+b$ were to be continued till the ſerieſes equal to ſuch higher powers were produced. Therefore the literal parts of the ſecond term, and of all the following terms of each of the ſaid products, or ſerieſes, may always be generated, or derived, from the firſt term of it, by the continual multiplication of it by the fraction $\frac{b}{a}$.

13. But, by the firſt obſervation, the firſt term of the ſeries which is equal to $\overline{a+b}|^m$, or the *m*th power of the binomial quantity $a+b$, when m is any whole number whatſoever, is a^m.

It follows therefore, in the 3d place, that the literal parts of the terms of the ſeries that is equal to $\overline{a+b}|^m$, will be a^m, $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, $a^{m-4}b^4$, $a^{m-5}b^5$, $a^{m-6}b^6$, $a^{m-7}b^7$, $a^{m-8}b^8$, $a^{m-9}b^9$, &c, till we come to the quantity $a^{m-m}b^m$, (or $a^0 b^m$, or $1 \times b^m$), or b^m.

And thus we have diſcovered the compoſition of the literal parts of the terms of the ſeries which is equal to $\overline{a+b}|^m$, as fully as can be deſired. And we have likewiſe diſcovered that the co-efficient of the firſt term, of which the literal part is a^m, is always 1, or that the firſt term of the ſaid ſeries is a^m itſelf, and not any multiple of a^m. Q. E. I.

14. In the 4th place it is evident that all the terms of every product, or ſeries, ariſing from the multiplication of the binomial quantity $a+b$ into itſelf, muſt be connected together by the ſign $+$, or added to each other. And conſequently the literal parts of the ſeries that is equal to $\overline{a+b}|^m$, will

will be $a^m + a^{m-1}b + a^{m-2}b^2 + a^{m-3}b^3 + a^{m-4}b^4 + a^{m-5}b^5 + a^{m-6}b^6 + a^{m-7}b^7 + a^{m-8}b^8 + a^{m-9}b^9 +$ &c. $+ b^m$.

Of the numeral co-efficients of the second and other following terms of the product, or series, that is equal to $\overline{a+b}|^m$.

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15. We come now to inquire into the numeral co-efficients of the fecond and other following terms of the product, or feries, which is equal to $\overline{a+b}|^m$, or the mth power of the binomial quantity $a+b$.

Now the numeral co-efficient of the fecond term of this feries will always be equal to m, or the index of the power to which the binomial quantity $a + b$ is to be raifed. This may be demonftrated in the manner following.

In raifing the feveral powers of the binomial quantity $a+b$ by the continual multiplication of that quantity into itfelf, in the manner above exemplified in art. 10, it is evident that the faid fecond term of every new product, or feries, that is equal to a new power of $a + b$, is always produced by adding the product of the multiplication of the firft term of the feries that is equal to the next lower power of $a + b$ (of which firft term we have feen that 1 is always the co-efficient) by b to the product of the multiplication of the fecond term of the faid foregoing feries by a; the effect of which addition is, to increafe the co-efficient of the fecond term of the new feries by an unit, or fo as to make it exceed the co-efficient of the fecond term of the foregoing feries by an unit. Thus, the fecond term, $2ab$, of the feries

$a^2 +$

$2ab + b^2$, which is equal to the square of $a+b$, is produced by the addition of the product ba, or ab, (which arises from the multiplication of a, the first term of the former series $a+b$, by b,) to the product ab, which arises from the multiplication of the second term b of the former series $a+b$ by a; the effect of which addition is, to make the co efficient, 2, of the second term $2ab$ in the new series, exceed the co-efficient, 1, of the second term b of the former series, by an unit. And, in like manner, the second term, $3a^2b$, of the series $a^3 + 3a^2b + 3ab^2 + b^3$, which is equal to the cube of the binomial quantity $a + b$, is produced by the addition of the product a^2b (which arises from the multiplication of a^2, the first term of the former series $a^2 + 2ab + b^2$, by b), to the product $2a^2b$, which arises from the multiplication of the second term $2ab$ of the said former series $a^2 + 2ab + b^2$ by a; the effect of which addition is, to make the co-efficient, 3, of the second term $3a^2b$ of the new series, exceed the co efficient, 2, of the second term $2ab$ of the former series, by an unit. And, in like manner, $4a^3b$, the second term of the next series, is $= a^3 \times b + 3a^2b \times a$, or $a^3b + 3a^3b = \overline{1 + 3}| \times a^3b$; and $5a^4b$, the second term of the next series, is $= a^4 \times b + 4a^3b \times a = a^4b + 4a^4b = \overline{1 + 4}| \times a^4b$; and $6a^5b$, the second term of the next series, is $= a^5 \times b + 5a^4b \times a = a^5b + 5a^5b = \overline{1 + 5}| \times a^5b$. And this, it is easy to see, must be the case in any higher powers whatsoever of $a+b$, if we were to continue the multiplications by $a+b$ till the serieses that were equal to such higher powers of $a+b$ were produced. And consequently, since in the first power of the binomial quantity $a+b$, to wit, in the said quantity itself, the co-efficient of the second term b, to wit, 1, is equal to the index of the said first power, which is also 1, and in the second, and third, and fourth, and fifth, and sixth powers of the said binomial quantity, the co efficient of the second term of the series that is equal to each of the said powers of $a+b$ is also equal to the index of the said powers; it follows that in all higher powers whatsoever of the said binomial quantity $a+b$, the co-efficient of the second term of the series which is equal to every such power

 will

will be equal to the index of the said power; or, in other words, the co-efficient of the second term of the series which is equal to $\overline{a+b}|^m$ will always be equal to the index m.

Q. E. D.

16. From what has been shewn in the foregoing articles we may conclude with certainty, that the two first terms of the series that is equal to $\overline{a+b}|^m$, when m is equal to any whole number whatsoever, will be $a^m + m \times a^{m-1}b$, and that the literal parts of the following terms of the said series will be $a^{m-2}b^2 + a^{m-3}b^3 + a^{m-4}b^4 + a^{m-5}b^5 + a^{m-6}b^6 + a^{m-7}b^7 + a^{m-8}b^8 + a^{m-9}b^9 + \&c. + a^{m-m}b^m$, (or $a^0 b^m$, or $1\ b^m$, or) $+\ b^m$. It remains that we inquire what will be the numeral co-efficients of the said third and other following terms of the said series, or by what law, or rule, they may be generated, or derived, from the two first co-efficients, 1 and m. This is a matter of considerable difficulty; and I am not acquainted with any direct and scientific method of investigating this law of the generation of the said co-efficients, besides that of Mr. James Bernoulli above mentioned, which is grounded on the Doctrine of Permutations and Combinations, and the properties of the Figurate numbers. But I can point out a manner of considering the subject and attempting to find this law of generation, which seems likely enough to have occurred to a Mathematician who was in pursuit of this inquiry, and which, if it had occurred to him, would have led him directly to form a just conjecture concerning this law by which these co-efficients are to be generated; after which he would have been induced to try the law, so discovered by conjecture, in some easy particular instances, and, having found it to be true in all of them, he would naturally conclude that it was true in all other cases whatsoever. This conjectural method of investigation, I conceive, may have been as follows.

A Con-

A Conjectural Investigation of the Law by which the co-efficients of the third, and fourth, and fifth, and other following terms of the series which is equal to $\overline{a+b}|^m$, *or the* m*th power of the binomial quantity* a + b, *may be generated, or derived, from* 1 *and* m, *the co-efficients of the two first terms of the said series.*

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17. Now, in order to discover the manner in which these co-efficients may be derived from the two first co-efficients 1 and m, I should think it would be natural to examine the co-efficients of the terms of the series that is equal to $\overline{a+b}|^m$ in some of the lower powers of $a+b$ which we have actually raised by multiplication, as, for example, in the series which is equal to $\overline{a+b}|^6$, and which we have found above in art. 10, to be $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$. In this series the co-efficients of the terms are 1, 6, 15, 20, 15, 6, and 1; and our object is to discover, 1st, by what number, integral or fractional, the second of these co-efficients, to wit, 6, ought to be multiplied in order to produce the third co-efficient, to wit, 15; and, 2dly, by what number, integral or fractional, the third co-efficient, to wit, 15, ought to be multiplied in order to produce the fourth co-efficient 20; and, 3dly, by what number the fourth co-efficient, 20, ought to be multiplied in order to produce the fifth co-efficient 15; and, 4thly, by what number the fifth co-efficient, 15, ought to be multiplied in order to produce the sixth co-efficient 6; and, lastly, by what number the sixth co-efficient, 6, ought to be multiplied in order to produce the seventh and last co-efficient 1. Now these multiplying numbers are evidently $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$. For $6 \times \frac{15}{6}$ is $= 15$, and $15 \times \frac{20}{15}$ is $= 20$, and $20 \times \frac{15}{20}$ is $= 15$,

and

and 15 × $\frac{6}{15}$ is = 6, and 6 × $\frac{1}{6}$ is = 1. We muſt therefore now examine theſe five generating fractions $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$, together with the preceeding generating fraction $\frac{6}{1}$, by the multiplication of which into the firſt co efficient 1 the ſecond co-efficient 6 is produced; and muſt endeavour to find out ſome remarkable, or regular, property in them, which we may reaſonably ſuppoſe to belong alſo to the co-efficients of the terms of other powers of $a+b$, as well as to thoſe of the terms of this, its ſixth power. And, in order to examine theſe fractions with the greater eaſe, it ſeems natural, in the firſt place, to reduce them to their loweſt denominations, by dividing both their numerators and their denominators by the factors which are common to them both. Now, if this be done, the ſaid generating fractions $\frac{6}{1}$, $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$, will be found to be equal to ($\frac{6}{1}$, $\frac{5\times 3}{2\times 3}$, $\frac{4\times 5}{3\times 5}$, $\frac{3\times 5}{4\times 5}$, $\frac{2\times 3}{5\times 3}$, and $\frac{1}{6}$, or) $\frac{6}{1}$, $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{1}{6}$; in which laſt fractions it is impoſſible not to obſerve that the numerators regularly decreaſe by an unit from 6, which is the index of the power to which the binomial quantity $a+b$ has been raiſed, to 1, and the denominators regularly increaſe at the ſame time by the ſame quantity of an unit from 1 to the ſaid index 6. This regularity is very ſtriking, and naturally raiſes a ſuſpicion that the ſame thing may take place in the generating fractions of the co-efficients of the terms of the ſerieſes that are equal to other powers of the binomial quantity $a+b$, and is an inducement to try it in the other ſerieſes that have been produced above in art. 10, by multiplication, and which are equal to $\overline{a+b}|^5$, $\overline{a+b}|^4$, $\overline{a+b}|^3$, and $\overline{a+b}|^2$. We will therefore now proceed to try it in thoſe inſtances.

18. Now

18. Now we have seen in art. 10, that $\overline{a+b}|^5$ is = the series $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, in which the numeral co-efficients of the terms are 1, 5, 10, 10, 5, and 1. Therefore the generating fractions, by the multiplication of which the second of these co-efficients, to wit, 5, is derived from the first, or 1, and every following co efficient from that which is next before it, will be $\frac{5}{1}$, $\frac{10}{5}$, $\frac{10}{10}$, $\frac{5}{10}$, and $\frac{1}{5}$; which are respectively equal to the fractions $\frac{5}{1}$, $\frac{4}{2}$, $\frac{3}{3}$, $\frac{2}{4}$, and $\frac{1}{5}$. And in these last fractions we cannot but observe that the numerators 5, 4, 3, 2, and 1, regularly decrease by an unit from 5, or the index of the power to which the binomial quantity $a+b$ is raised, to 1, and the denominators 1, 2, 3, 4, and 5, regularly increase at the same time, by the same quantity of an unit from 1 to the said index 5. It appears therefore that the same rule takes place amongst these generating fractions $\frac{5}{1}$, $\frac{4}{2}$, $\frac{3}{3}$, $\frac{2}{4}$, and $\frac{1}{5}$, as took place amongst the generating fractions $\frac{6}{1}$, $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{1}{6}$, of the co-efficients of the terms of the former series which was equal to $\overline{a+b}|^6$.

19. We will now try whether the same rule will take place in the series which is equal to the fourth power of $a+b$.

This series is $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, in which the numeral co efficients of the terms are 1, 4, 6, 4, and 1. Now the generating fractions by the multiplication of which the second of these co-efficients, to wit, 4, is generated from the first, or 1, and every following co-efficient is generated from that which is next before it, are evidently $\frac{4}{1}$, $\frac{6}{4}$, $\frac{4}{6}$,

and

and $\frac{1}{4}$; which are respectively equal to $\frac{4}{1}$, $\frac{3}{2}$, $\frac{2}{3}$, and $\frac{1}{4}$. And in these last fractions the numerators 4, 3, 2, and 1, regularly decrease by an unit from 4, which is the index of the power to which the binomial quantity $a+b$ has been raised, to 1, and the denominators 1, 2, 3, and 4, regularly increase at the same time by an unit from 1 to the said index 4.

20. We come next to the series which is equal to the cube of $a+b$.

This series is $a^3 + 3a^2b + 3ab^2 + b^3$, in which the co-efficients of the terms are 1, 3, 3, and 1. Now the generating fractions, by the multiplication of which the second of these co-efficients is derived from the first, and the third from the second, and the fourth from the third, are evidently $\frac{3}{1}$, $\frac{3}{3}$, and $\frac{1}{3}$, which are respectively equal to $\frac{3}{1}$, $\frac{2}{2}$, and $\frac{1}{3}$. And in these last fractions the numerators 3, 2, and 1, decrease regularly by an unit from 3, which is the index of the power to which the binomial quantity $a+b$ has been raised, to 1, and the denominators 1, 2, and 3, increase regularly at the same time by an unit from 1 to the said index 3.

21. And the same thing takes place in the series which is equal to the square of $a+b$. For this series is $a^2+2ab+b^2$, in which the co-efficients of the terms are 1, 2, and 1. Now the generating fractions, by the multiplication of which the second co-efficient 2 is derived from the first co-efficient 1, and the third co-efficient 1 is derived from the second co-efficient 2, are evidently $\frac{2}{1}$, and $\frac{1}{2}$; which admit of no reduction to lower denominations. And in these fractions $\frac{2}{1}$ and $\frac{1}{2}$, the numerators 2 and 1 decrease by an unit, as

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in the former cafes, from 2, which is the index of the power to which the binomial quantity $a + b$ has been raifed, to 1, and the denominators 1 and 2 increafe at the fame time by an unit from 1 to the faid index 2.

22. It appears therefore that this law of the generating fractions of the numeral co-efficients of the terms of the feriefes that are equal to the powers of the binomial quantity $a + b$, takes place in the cafes of the fquare, the cube, the fourth power, the fifth power, and the fixth power, of the faid binomial quantity. This is a very ftrong ground for conjecturing that the fame law will take place in the generating fractions of the numeral co-efficients of the terms of the feriefes which are equal to the powers of the faid binomial quantity in all other cafes whatfoever; or that, if the index of the power to which the faid binomial quantity is raifed be any whole number whatfoever, denoted by the letter m, the generating fractions, by the continual multiplication of which the numeral co-efficients of the fecond and other following terms of the feries which is equal to $\overline{a + b}|^m$, or the mth power of the faid binomial quantity, may be derived from 1, or the co-efficient of the firft term, a^m, or $1 \times a^m$, of the faid feries, will be $\frac{m}{1}$, or (as it is fometimes called) $\frac{m-0}{1}$, and $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, till we come to the term $\frac{m-m}{m+1}$, which is $= 0$, or till the faid feries is terminated, or exhaufted.

23. And the ground for conjecturing that this is a general law that takes place among the generating fractions of the terms of thefe feriefes in all cafes, or when the index is equal to any whole number whatfoever, will become ftill ftronger if we try it in a few more examples of feriefes that are equal to higher powers of the binomial quantity $a + b$, than the fixth power. I fhall therefore now proceed to try it

it in the seriefes which are equal to $\overline{a+b}|^7$, $\overline{a+b}|^8$, and $\overline{a+b}|^9$.

24. Now $\overline{a+b}|^7$ is $(=\overline{a+b}|^6 \times \overline{a+b} = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \times \overline{a+b}) = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$; in which series the numeral co-efficients of the terms are 1, 7, 21, 35, 35, 21, 7, and 1. The generating fractions, by the multiplication of which the second of these co efficients, to wit, 7, is derived from the first co-efficient 1, and the third and other following co-efficients are derived from those which immediately preceed them, are evidently $\frac{7}{1}$, $\frac{21}{7}$, $\frac{35}{21}$, $\frac{35}{35}$, $\frac{21}{35}$, $\frac{7}{21}$, and $\frac{1}{7}$; which are respectively equal to $\frac{7}{1}$, $\frac{6}{2}$, $\frac{5}{3}$, $\frac{4}{4}$, $\frac{3}{5}$, $\frac{2}{6}$, and $\frac{1}{7}$; in which last fractions the numerators 7, 6, 5, 4, 3, 2, and 1, regularly decrease by an unit from 7, (which is the index of the power to which the binomial quantity $a+b$ has been raised), to 1, and the denominators 1, 2, 3, 4, 5, 6, and 7, regularly increase at the same time by an unit from 1 to the said index 7; agreeably to what was observed in the five former examples.

25. And $\overline{a+b}|^8$ is $(=\overline{a+b}|^7 \times \overline{a+b} = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 \times \overline{a+b}) = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$; in which series the numeral co-efficients of the terms are 1, 8, 28, 56, 70, 56, 28, 8, and 1. The generating fractions of these co-efficients are evidently $\frac{8}{1}$, $\frac{28}{8}$, $\frac{56}{28}$, $\frac{70}{56}$, $\frac{56}{70}$, $\frac{28}{56}$, $\frac{8}{28}$, and $\frac{1}{8}$; which are respectively equal to $\frac{8}{1}$, $\frac{7}{2}$, $\frac{6}{3}$, $\frac{5}{4}$, $\frac{4}{5}$, $\frac{3}{6}$, $\frac{2}{7}$, and $\frac{1}{8}$; in which last fractions the numerators 8, 7, 6, 5, 4, 3, 2, and 1, decrease regularly by an unit from 8, (which is the index of the power to

2 which

which the binomial quantity $a+b$ has been raised), to 1, and the denominators 1, 2, 3, 4, 5, 6, 7, and 8, regularly increase at the same time by an unit from 1 to the said index 8; agreeably to what has been observed in the six former examples.

26. And, lastly, $\overline{a+b}|^9$ is $(= \overline{a+b}|^8 \times \overline{a+b} = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8 \times \overline{a+b}) = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$; in which series the numeral co-efficients of the terms are 1, 9, 36, 84, 126, 126, 84, 36, 9, and 1. The generating fractions of these co-efficients are evidently $\frac{9}{1}$, $\frac{36}{9}$, $\frac{84}{36}$, $\frac{126}{84}$, $\frac{126}{126}$, $\frac{84}{126}$, $\frac{36}{84}$, $\frac{9}{36}$, and $\frac{1}{9}$; which are respectively equal to $\frac{9}{1}$, $\frac{8}{2}$, $\frac{7}{3}$, $\frac{6}{4}$, $\frac{5}{5}$, $\frac{4}{6}$, $\frac{3}{7}$, $\frac{2}{8}$, and $\frac{1}{9}$; in which last fractions the numerators 9, 8, 7, 6, 5, 4, 3, 2, and 1, decrease regularly by an unit from 9, (which is the index of the power to which the binomial quantity $a+b$ has been raised), to 1, and the denominators 1, 2, 3, 4, 5, 6, 7, 8, and 9, regularly increase at the same time by an unit from 1 to the said index 9; agreeably to what has been observed in all the former examples.

27. After observing this law of the co-efficients to take place in so many different examples, it would be impossible for our mathematical investigator not to conclude with a very high degree of confidence that it would take place in all other cases whatsoever; or that, when the index m is equal to any whole number whatsoever, the generating fractions of the numeral co-efficients of the terms of the series that is equal to $\overline{a+b}|^m$, will be $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, &c, till we come to the fraction $\frac{m-m}{m+1}$, which is

2 K = 0,

$= 0$, or till the ſaid ſeries of fractions is terminated, or exhauſted. And then it would follow, from what has been ſhewn above concerning the literal parts of the terms of the ſaid ſeries, that the ſaid ſeries which is equal to $\overline{a+b}|^m$, or the mth power of the binomial quantity $a+b$, would be $a^m + \frac{m}{1} \text{A} a^{m-1} b + \frac{m-1}{2} \text{B} a^{m-2} b^2 + \frac{m-2}{3} \text{C} a^{m-3} b^3 + \frac{m-3}{4} \text{D} a^{m-4} b^4 + \frac{m-4}{5} \text{E} a^{m-5} b^5 + \frac{m-5}{6} \text{F} a^{m-6} b^6 +$ &c, continued to b^m. Q. E. I.

28. This method of diſcovering (by a conjecture grounded on ſome trials in particular examples) that the generating fractions by which the numeral co-efficients of the third, and fourth, and other following terms of the ſeries that is equal to $\overline{a+b}|^m$ (or any integral power of the binomial quantity $a+b$), are derived from m (the index of the power to which the ſaid binomial quantity is raiſed), or from the co-efficient of the ſecond term of the ſaid ſeries (which is always equal to the ſaid index) are $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, is ſuggeſted by Profeſſor Saunderſon, in the ſecond volume of his Algebra, in the chapter on the Binomial Theorem; where the Reader will find a good explanation and illuſtration of the ſaid celebrated Theorem, by a variety of examples, both in the caſe of Integral powers, and in the caſe of Roots and other Fractional powers, and even in the caſe of Negative powers, and of powers that are both fractional and negative; but no demonſtration of it in any caſe, not even in that of Integral and Affirmative powers.

29. We have now ſhewn with demonſtrative certainty that the literal parts of the terms of the ſeries which is equal to $\overline{a+b}|^m$, or the mth power of the binomial quantity $a+b$, when

when the letter m denotes any affirmative whole number whatſoever, are $a^m + a^{m-1}b + a^{m-2}b^2 + a^{m-3}b^3 + a^{m-4}b^4 + a^{m-5}b^5 + a^{m-6}b^6 + a^{m-7}b^7 +$ &c, till we come to the term $a^{m-m}b^m$, (or $a^0 \times b^m$, or $1 \times b^m$), or b^m, and that the numeral co-efficient of the literal part of the firſt term of the ſaid ſeries is 1, and the numeral co-efficient of the literal part of the ſecond term of it is m, and conſequently that the two firſt whole terms of the ſaid ſeries are $1 \times a^m$, and $m \times a^{m-1}b$, or $a^m + m \times a^{m-1}b$, or $a^m + \frac{m}{1} \times a^{m-1}b$, or $a^m + \frac{m-0}{1} \times a^{m-1}b$. And we have alſo ſhewn that in the ſerieſes which are equal to $\overline{a+b}|^2$, $\overline{a+b}|^3$, $\overline{a+b}|^4$, $\overline{a+b}|^5$, $\overline{a+b}|^6$, $\overline{a+b}|^7$, $\overline{a+b}|^8$, and $\overline{a+b}|^9$, or when the index m is equal to 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the ſaid ſerieſes are derived from m, or the numeral co-efficient of the ſecond term, by the continual multiplication of the fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, $\frac{m-8}{9}$, and $\frac{m-9}{10}$; which affords a very ſtrong ground for conjecturing that the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the ſerieſes that are equal to any higher powers whatſoever of the binomial quantity $a+b$, will, in like manner, be derived from m, the co-efficient of the ſecond term, by the continual multiplication of the ſame generating fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, $\frac{m-6}{7}$, $\frac{m-7}{8}$, $\frac{m-8}{9}$, $\frac{m-9}{10}$, $\frac{m-10}{11}$, $\frac{m-11}{12}$, &c, till we come to the term $\frac{m-m}{m+1}$, which is $= 0$, or till the ſaid ſeries of generating fractions is terminated, or exhauſted. Now this con-

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jecture may be changed into absolute certainty, and the said law of the generation of the co-efficients may be shewn to take place in all the serieses that are equal to the quantity $\overline{a+b}|^m$, or the powers of the binomial quantity $a+b$, when the index m of the said quantity $\overline{a+b}|^m$, is equal to any whole number, how great soever, by shewing that, if it takes place when the index m is equal to any one particular number, (as we have seen that it does when the index m is equal either to 2, or to 3, or to 4, or to 5, or to 6, or to 7, or to 8, or to 9), it must also take place when the index m is greater by an unit than in the former case. For then it will follow that it must be true likewise when the index m is greater by any multitude of units than in the former case, or when it is equal to any other whole number, how great soever. This we shall now proceed to shew in the remaining part of this discourse.

Of the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the Series that is equal to $\overline{a+b}|^m$, *and the law of the generation of the said co-efficients from* m, *the co-efficient of the second term of the said series, and from each other.*

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30. In order to demonstrate the law of the generation of these co-efficients, it will be convenient to get rid of the powers of a and b, in the terms of the series that is equal to $\overline{a+b}|^m$, and to fix our attention only on the generation of the numeral co-efficients of the third, fourth, fifth, sixth, and other following terms of the said series. This may be done by supposing a and b to be, each of them, equal to 1,

and consequently $a+b$ to be equal to $1+1$, and $\overline{a+b}|^m$ to be equal to $\overline{1+1}|^m$. For, as all the powers of both a and b will, on this supposition, be equal to 1, the Binomial Theorem set forth above in art. 1, will then be reduced to this, to wit, that $\overline{1+1}|^m$ will be equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ + &c, continued to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ × &c, $\times \frac{m-\overline{m-1}}{m}$, or to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ × &c, $\times \frac{m-m+1}{m}$, or to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ × &c, $\times \frac{1}{m}$, or to the term 1. For the last term of this series must always be 1; because the numerators of the several factors in it form a decreasing progression of numbers, decreasing by an unit, from m to 1; and the denominators of the same factors form an increasing progression of numbers, increasing by an unit, from 1 to m; and consequently the product of the multiplication of all the denominators is equal to the product of the multiplication of all the numerators, and therefore the product of the multiplication of all the said factors, or fractions, $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, &c, into each other, or the last term of the series, must always be equal to 1.

We are therefore now to demonstrate that $\overline{1+1}|^m$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1}$ ×

$\times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ + &c, + 1. And this we propose to do by shewing, by abstract and general reasonings, that, if this Theorem is true when the index m is of any particular value, as, for example, when it is equal to 9, it must likewise be true when the index m is increased by an unit, or that, if n be taken $= m+1$, the quantity $\overline{1+1}|^n$, or the nth power of the binomial quantity $1+1$, will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$ + &c, continued to the term $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times$ &c, $\times \frac{1}{n}$, or to the term 1.

31. To facilitate the demonstration of this proposition, it will be convenient to premise the following Lemma.

A LEMMA.

If the terms of the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ + &c, + 1 (in which m represents any whole number whatsoever), be set down twice together in two parallel lines, or rows, one under the other, but with the terms in the lower row advanced one step further to the right-

right-hand than the terms in the upper row, ſo that the firſt term in the lower row ſhall ſtand under the ſecond term of the upper row, and the ſecond term in the lower row ſhall ſtand under the third term in the upper row, and the third, fourth, fifth, ſixth, and other following terms in the lower row ſhall ſtand under the fourth, fifth, ſixth, ſeventh, and other following terms in the upper row, reſpectively; and both rows are continued to the ſame number of terms, namely, to the whole number of terms in the ſaid ſeries, or to $m+1$ terms; and then the terms in the lower row (each of which, it is evident, will conſiſt of one factor leſs than the correſponding term, or term ſtanding immediately above it in the upper row) be reduced to the ſame denomination as the terms that ſtand immediately above them in the upper row, and, after being ſo reduced, are added to the ſaid terms that ſtand immediately above them in the ſaid upper row;—upon theſe ſuppoſitions the new ſeries of terms ariſing from this addition of the ſaid two rows of terms to each other, will be as follows, to wit, $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m+1}{5} + \&c + 1$; in which ſeries the laſt term is 1, as well as in the two ſerieſes from the addition of which this ſeries ariſes; and the numerators of the laſt factors in all the terms, except the laſt, are always equal to $m+1$, inſtead of being equal to $m-1$, $m-2$, $m-3$, $m-4$, &c, as in the two foregoing ſerieſes; and the number of terms in the ſaid new ſeries is $m+2$, inſtead of $m+1$, which is the number of terms in each of the ſaid foregoing ſerieſes.

DEMON-

DEMONSTRATION.

32. This will appear by setting down the said series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c, + 1 twice over, in the manner that has been just described; which may be done as follows:

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c.$$

$$1 + \frac{m}{1} \qquad + \frac{m}{1} \times \frac{m-1}{2} \qquad + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \qquad + \&c.$$

In these two rows of terms it is evident, in the first place, that the terms in the upper row, after the two first terms 1 and $\frac{m}{1}$, consist of two, three, and four, and more, factors, every new term having one more factor than the term next before it; and, 2dly, that the terms in the lower row that stand immediately under the third, fourth, fifth, and other following terms in the upper row, consist of one factor less than the corresponding terms, or terms immediately over them in the upper row; and, 3dly, that the terms in the lower row consist of the very same factors as the corresponding terms in the upper row, excepting that they want the last factors of the said terms in the upper row. And hence it follows, that, in order to reduce the terms in the lower row to the same denomination as the terms in the upper row, we must multiply them by factors that shall have the same denominators as the last, or additional factors in the upper row, and which must have their numerators equal to their denominators, so as to make each of them equal to 1, to the end that the magnitudes of the said lower terms may not be altered by the multiplication of them by the said new factors. Thus, for example, the second term of the lower

lower row, to wit, $\frac{m}{1}$, muſt be multiplied into the factor $\frac{2}{2}$, in order to bring it to the ſame denomination as the third term in the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2}$, without altering its magnitude; and the third term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2}$, muſt be multiplied into the factor $\frac{3}{3}$, in order to bring it to the ſame denomination as the fourth term of the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, without altering its magnitude; and the fourth term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, muſt be multiplied into the factor $\frac{4}{4}$, in order to bring it to the ſame denomination as the fifth term in the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, without altering its magnitude; and, for the like reaſon, the fifth, and ſixth, and ſeventh, and other following terms in the lower row muſt be multiplied into the ſeveral factors $\frac{5}{5}$, and $\frac{6}{6}$, and $\frac{7}{7}$, &c, reſpectively; after which multiplications the two rows of terms that are to be added to each other, will be as follows, to wit,

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$$

$$+ 1 + \frac{m}{1} \times \frac{2}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{3}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{4}{4} + \&c.$$

53. And, if theſe two rows of terms (being now brought to the ſame denominations) are added together in the manner above deſcribed; that is, every term in the lower row to the term that is immediately above it, the ſum thence reſulting will be the ſeries $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c$, in which the

the numerator of the laſt factor in every term is always $m + 1$, inſtead of $m - 1$, $m - 2$, $m - 3$, $m - 4$, &c.

And "That this muſt be the caſe in all the following "terms of the ſaid new ſeries as well as in the few terms of "it that have been here ſet down," will be evident from this conſideration, to wit, That the denominator of the laſt factor of every term in the upper of the two rows of terms that are added together is always greater by an unit than the number which is ſubtracted from m in the numerator of the ſame factor. For from thence it follows that the denominator of the new multiplying fraction in the correſponding term of the lower row (which is always equal to the denominator of the ſaid laſt factor in the upper row,) muſt always be greater by an unit than the number which is ſubtracted from m in the numerator of the laſt factor of the ſaid upper term. And, therefore, the numerator of the ſaid new multiplying fraction in the lower row (which is always equal to its denominator,) muſt alſo always be greater by an unit than the number which is ſubtracted from m in the numerator of the laſt factor of the ſaid upper term; the conſequence of which, in adding the lower term to the upper term, is to convert the numerator of the laſt factor in the upper term from $m - 1$, or $m - 2$, or $m - 3$, or the exceſs of m above ſome other number, into $m + 1$. Q. E. D.

34. And the number of terms in the new ſeries, ariſing from the addition of the two former in the manner that has been deſcribed, will be greater by one than the number of the terms in either of the two added ſerieſes: becauſe the lower row of terms, conſiſting of the ſame number of terms as the upper row, and being placed one term further to the right-hand, muſt extend one term beyond it; and conſequently, as the number of terms in each of the two rows of terms is $m + 1$, the number of terms in the new ſeries, ariſing from the addition of the two rows together, muſt be $m + 2$. Q. E. D.

35. And, laſtly, the laſt term of the ſaid new ſeries muſt be the ſame as the laſt term of the old ſeries, or of the lower row

row of terms; becaufe, as the lower row of terms extends one term beyond the upper row, the laft term in the lower row will not have any term over it in the upper row to which it is to be added, and confequently will continue the fame in the new feries $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} +$ &c, as in the old feries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c. But we have feen above, in art. 30, that the laft term of the feries $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c, is 1. Therefore the laft term in the new feries $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} +$ &c, will alfo be 1. Q. E. D.

36. Coroll. 1. Now let the order of the numerators m, $m-1$, $m-2$, $m-3$, $m-4$, &c, and $m+1$, of the factors of the third and other following terms of the laft feries $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} +$ &c, be changed, by making $m+1$ the numerator of the firft factor of every term inftead of being the numerator of the laft factor. The faid feries will then be as follows, to wit, $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} +$ &c. Now this change in the order of the numerators of the feveral factors of the terms will create no change in the values,

 or

or magnitudes, of the ſeveral terms themſelves; becauſe the products ariſing from the multiplication of the ſame numbers are always the ſame, in whatever order the numbers are multiplied. Therefore the foregoing ſeries, after this change in the order of the numerators of the ſeveral factors of its terms, will ſtill be of the ſame magnitude as before, and conſequently will be equal to the ſum that ariſes from the addition of the aforeſaid two rows of terms in the manner above deſcribed; that is, the ſeries $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} +$ &c, $+ 1$, will be equal to the ſum that ariſes from the addition of the aforeſaid two rows of terms in the manner above deſcribed.

37. Coroll. 2. Now let n be $= m + 1$. Then will $n-1$ be $= m$, and $n - 2$ will be $= m - 1$, and $n-3$ will be $=$ $m - 2$, and $n - 4$ will be $= m - 3$; and, in like manner, $n - 5$, $n - 6$, $n - 7$, &c, will be equal to $m - 4$, $m - 5$, $m - 6$, &c, reſpectively. And conſequently the ſeries obtained in the foregoing Corollary, to wit, $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4}$ $+$ &c, $+ 1$, conſiſting of $m+2$ terms, will be equal to the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c, $+ 1$, conſiſting of $n+1$ terms. Therefore the ſeries $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c, $+ 1$, conſiſting of $n + 1$ terms, will be equal to the ſum that ariſes by adding the two aforeſaid rows of terms together in the manner above deſcribed.

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The Demonstration of the principal Proposition.

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38. These things being premised, the main proposition stated at the end of art. 30, to wit, that, if m denote any whole number whatsoever, the quantity $\overline{1+1}|^m$, or the mth power of the binomial quantity $1+1$, will be equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ + &c, continued to $m+1$ terms, or to the term 1, may be demonstrated in the manner following.

39. The product that arises by multiplying the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m-3}{4}$ + &c, into $1+1$ is the sum that arises by setting down the said series twice following in two parallel rows, one under the other, with the terms in the lower row advanced one term further to the right-hand than the terms in the upper row, in the manner above described, and then adding the terms in the lower row to the corresponding terms in the upper row. And the $\overline{m+1}|$th power of $1+1$ is the product of the multiplication of the mth power of $1+1$ into $1+1$. Therefore, if in any particular value of m the mth power of $1+1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ + &c, + 1, consisting of $m+1$ terms, the $\overline{m+1}|$th power of $1+1$ will be equal to the sum that arises by setting down the said series twice following in two parallel rows in the manner above described, and adding the said two rows of terms

terms together. But, by the second Corollary of the foregoing Lemma, if n be $= m+1$, the sum arising from the addition of the said two rows of terms is the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c, $+ 1$, consisting of $n+1$ terms. Therefore, if in any particular value of m the mth power of $1+1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c, $+ 1$, consisting of $m+1$ terms, it will follow that the $\overline{m+1}|$th, or nth, or next higher power, of $1+1$ will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c $+ 1$, consisting of $n+1$ terms. But it has been shewn in art. 17, 18, 19, &c, - - - 26, that when m is equal either to 2, or to 3, or to 4, or to 5, or to 6, or to 7, or to 8, or to 9, the mth power of $1+1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c $+ 1$, consisting of $m+1$ terms. Therefore, if n be equal to $9+1$, or 10, the $\overline{9+1}|$th power, or 10th power, or nth power, of $1+1$ will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$ &c $+ 1$, consisting of $n+1$, or $10+1$, or 11, terms. And in the same manner it may be proved that, since, when m is $= 10$, the mth power of $1+1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c, $+ 1$, consisting of $m+1$, or $10+1$, or 11, terms, the $\overline{m+1}|$th, or or $\overline{10+1}|$th, or 11th, or (putting $n = m+1 = 10+1 = 11$) the nth, power of $1+1$ will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ + &c + 1, consisting of $n+1$, or $11+1$, or 12, terms. And so we may proceed from number to number *ad infinitum*. And consequently, whatever be the whole number denoted by m, it will always be true that $\overline{1+1}|^m$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$ + &c + 1, consisting of $m+1$ terms. Q. E. D.

The foregoing Demonstration expressed in a more concise Manner.

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40. The foregoing reasonings may be expressed in a more concise manner as follows. If n be $= m+1$, and it be true in any particular value of m that $\overline{1+1}|^m$ is = the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ + &c, it will also be true that $\overline{1+1}|^n$ will be $= 1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ + &c.

For $\overline{1+1}|^n$ is $= \overline{1+1}|^{m+1} = \overline{1+1}|^m \times \overline{1+1}$ = the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2}$
×

$\times \frac{m-2}{3} \times \frac{m-3}{4} +$ &c, multiplied into $1+1 =$

$$1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+\&c.$$

$$+1+\frac{m}{1} \qquad +\frac{m}{1}\times\frac{m-1}{2} \qquad +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3} \qquad +\&c.$$

$$=1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+\&c.$$

$$+1+\frac{m}{1}\times\frac{2}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{3}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{4}{4}+\&c.$$

$$=1+\frac{m+1}{1}+\frac{m}{1}\times\frac{m+1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m+1}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m+1}{4}+\&c$$

$$=1+\frac{m+1}{1}+\frac{m+1}{1}\times\frac{m}{2}+\frac{m+1}{1}\times\frac{m}{2}\times\frac{m-1}{3}+\frac{m+1}{1}\times\frac{m}{2}\times\frac{m-1}{3}\times\frac{m-2}{4}+\&c$$

$$=1+\frac{n}{1}+\frac{n}{1}\times\frac{n-1}{2}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}+\&c.$$

But it has been ſhewn in art. 17, 18, 19, &c, - - - - 26, that, when m is equal either to 2, or to 3, or to 4, or to 5, or to 6, or to 7, or to 8, or to 9, $\overline{1+1}^{m}$ is equal to the ſeries $1+\frac{m}{1}+\frac{m}{1}\times\frac{m-1}{2}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}+$ &c. Therefore, if n be $= 9+1$, or 10, $\overline{1+1}|^{n}$, or $\overline{1+1}|^{10}$, will be = the ſeries $1+\frac{n}{1}+\frac{n}{1}\times\frac{n-1}{2}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$ + &c. And it may be ſhewn in like manner, that, if n be put for 11, 12, 13, 14, &c, *ad infinitum* ſucceſſively, $\overline{1+1}|$ will, in all theſe ſuppoſitions, be always equal to the ſeries $1+\frac{n}{1}+\frac{n}{1}\times\frac{n-1}{2}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}+$ &c; and therefore the propoſition is univerſally true, whatever be the whole number denoted by the letter n.

Q. E. D.

41. This

41. This demonstration of the binomial theorem in the case of integral powers, is nearly the same with that given by Mr. John Stewart, of Aberdeen, in the 6th Section of his Commentary on Sir Isaac Newton's curious little Tract, intitled, *Analysis by Equations of an infinite number of Terms.* See his edition of Newton's Treatise *on the Quadrature of Curves*, and of the said Tract intitled *Analysis, &c*, with his learned Comments on both, in one volume, quarto, published at London, in the year 1745, page 471, Art. 155.

Of the Powers of a Residual Quantity $a - b$, when their Indexes are whole Numbers.

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42. We have hitherto been considering the integral powers of a *binomial* quantity $a + b$, or of the *sum* of two single quantities a and b; and we have seen that, if the said binomial quantity $a+b$ be raised to any power of which a whole number denoted by m is the index, the quantity $\overline{a+b}|^m$, or the said mth power of $a + b$, will be equal to the series $a^m + \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c, + b^m$, or (if we put $A = 1$, $B = \frac{m}{1} A$, $C = \frac{m-1}{2} B$, $D = \frac{m-2}{3} C$, $E = \frac{m-3}{4} D$, $F = \frac{m-4}{5} E$, and $G, H, I, K, L, \&c, = \frac{m-5}{6} F, \frac{m-6}{7} G, \frac{m-7}{8} H, \frac{m-8}{9} I, \frac{m-9}{10} K, \&c$, respectively,) to the series $a^m + \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 + \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 + \frac{m-4}{5} E a^{m-5} b^5 + \&c + b^m$; in

in which all the terms after the first term a^m are marked with the sign +, or are added to the said first term. We will now proceed to consider the value of $\overline{a-b}|^m$, or the *m*th power of the *residual* quantity $a - b$, or of the *difference* of the two quantities a and b, upon a supposition that a is the greater of the two.

43. Now, if a be supposed to be greater than b, and m be any whole number whatsoever, the quantity $\overline{a-b}|^m$, or the *m*th power of the residual quantity, or difference, $a-b$, will be equal to the series $a^m - \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5$ + &c, or (if we put A, as before, = 1, and B = $\frac{m}{1}$ A, and C = $\frac{m-1}{2}$ B, and D = $\frac{m-2}{3}$ C, and E = $\frac{m-3}{4}$ D, and F = $\frac{m-4}{5}$ E. and G, H, I, K, L, &c, = $\frac{m-5}{6}$ F, $\frac{m-6}{7}$ G, $\frac{m-7}{8}$ H, $\frac{m-8}{9}$ I, $\frac{m-9}{10}$ K, &c, respectively,) to the series $a^m - \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 - \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 - \frac{m-4}{5} E a^{m-5} b^5$ + &c, which consists of exactly the same terms as the series that is equal to $\overline{a+b}|^m$, or the same power of the binomial quantity $a+b$, but with the sign — prefixed to the second, and fourth, and sixth, and every following even term in the series, which denotes that the said terms are not to be added to the first term a^m, and to the third, and fifth, and other following odd terms, (as they were in the former series, which was equal to $\overline{a+b}|^m$,) but to be subtracted from them.

44. That

44. That this must be so, will be evident from considering the manner in which the several powers of the residual quantity $a - b$ are generated from each other by the continual multiplication of $a—b$, of which we will now exhibit a specimen with respect to a few of its lowest powers. The second, third, fourth, and fifth powers of $a—b$ are derived from $a—b$ itself by the following multiplications.

$$\begin{array}{l}
a-b \\
a-b \\
\hline
aa-ab \\
\quad -ab+bb \\
\hline
aa-2ab+bb = \overline{a-b}|^2. \\
\qquad a-b \\
\hline
a^3-2a^2b+ab^2 \\
\quad -a^2b+2ab^2-b^3 \\
\hline
a^3-3a^2b+3ab^2-b^3 = \overline{a-b}|^3. \\
\qquad\qquad a-b \\
\hline
a^4-3a^3b+3a^2b^2-ab^3 \\
\quad -a^3b+3a^2b^2-3ab^3+b^4 \\
\hline
a^4-4a^3b+6a^2b^2-4ab^3+b^4 = \overline{a-b}|^4 \\
\qquad\qquad\qquad a-b \\
\hline
a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\
\quad -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5 \\
\hline
a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5 = \overline{a-b}|^5.
\end{array}$$

45. From these operations it is evident that, wherever the odd powers of b occur in the said powers of $a — b$, the terms are marked with the sign —, and that, wherever the even powers of b occur in the said powers of $a — b$, the terms are marked with the sign +. And the same thing, it is evident, must happen in all higher powers of $a—b$ whatsoever, as well in those that have been here set down, because

b is marked with the ſign — in the two original factors $a - b$ and $a - b$; whence it follows, from the nature of algebraick multiplication, that, whenever b is multiplied into itſelf an even number of times, the product will be marked with the ſign + ; and, whenever it is multiplied into itſelf an odd number of times, the product will be marked with the ſign —. And it is further evident, from the foregoing multiplications, that the odd powers of b occur in the ſecond, and fourth, and ſixth, terms of the foregoing products, and that the even powers of b occur in the third and fifth terms of them. And it is eaſy to ſee that the odd powers of b will occur in like manner in the eighth, and tenth, and twelfth, and other following even terms of all higher powers of $a - b$ whatſoever, and that the even powers of b will occur in like manner in the ſeventh, and ninth, and eleventh, and other following odd terms of the ſaid higher powers of $a - b$. And it is alſo evident, from the foregoing multiplications, that the terms themſelves of which the ſeveral powers of $a - b$ will be compoſed, are exactly the ſame with the terms of which the ſame powers of $a + b$ are compoſed. And hence it follows that the ſeries which is equal to $\overline{a - b}|^m$ will be the ſame with the ſeries which is equal to $\overline{a + b}|^m$, when the ſign — has been prefixed to the ſecond, and fourth, and ſixth, and other following even terms of it, inſtead of the ſign +, or that $\overline{a - b}|^m$, or the mth power of the reſidual quantity $a - b$, will be equal to the ſeries $a^m - \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 +$ &c, or $a^m - \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 - \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 - \frac{m-4}{5} E a^{m-5} b^5 +$ &c.

Q. E. D.

A DIS-

A

DISCOURSE

OF

COMBINATIONS,

ALTERNATIONS,

AND

ALIQUOT PARTS.

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OF

COMBINATIONS,

ALTERNATIONS,

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CHAP. I.

Of the variety of Elections, or Choice, in taking or leaving One or more, out of a certain Number of things proposed.

FOR the better underſtanding of what is propoſed; ſuppoſe we a certain number of counters or other things expoſed; as, for inſtance, 7; *a b c d e f g*: The queſtion is, what variety, or how many caſes there may be, of taking from thence one, or two of them; as *a*, *b*, *c*, *d*, &c. Or, *ab*, *ac*, *ad*, *bc*, *bd*, &c. Or, three's, as *abc*, *abd*, *acd*, *bdc*, &c. Or, fours, fives, &c. Or all, or none? And the like if any other number of things were ſo expoſed.

In order to the ſolution whereof, I ſhall here inſert a table, borrowed from my Arithmetick of Infinites, Prop. 132, 169, 183, 189, &c, (becauſe there will be often occaſion of having recourſe to it.) And then proceed to propoſitions thereunto relating.

To

To be left.

Numbers.	0	1	2	3	4	5	6	7	8	9	10	To be taken.
Monadicks.	1	1	1	1	1	1	1	1	1	1	1	0
Laterals.	1	2	3	4	5	6	7	8	9	10		1
Triangulars.	1	3	6	10	15	21						2
Pyramidals.	1	4	10	20	35							3
Triang. Triang.	1	5	15	35								4
Triang. Pyram.	1	6	21									5
Pyram. Pyram.	1	7										6
&c.	1	8										7
	1	9										8
	1	10										9
	1											10

Now, as to the conſtruction of this table, we are to obſerve, that, (the firſt line being all units,) the following numbers are, in every place, the aggregate of all thoſe in the line next above it, ſo far. As for example; for the three firſt in the uppermoſt line, 1, 1, 1, we have in the ſecond line (under the laſt of them) the number 3, which is the aggregate of them. And, in like manner, we have in the next place 4, which is the aggregate of 1, 1, 1, 1. (And ſo of the reſt.) And, in the lines following, likewiſe: So for 1, 2, 3, (the three foremoſt of the ſecond line,) we have in the third line (under the laſt of them) the number 6, equal to all of them: and ſo every where. This premiſed, the propoſitions follow.

1. It is manifeſt, that, if we would *take none*, that is, if we would *leave all*, there can be but one caſe thereof, whatever be the number of things expoſed. (For this admits of no variety.) Which (in the table) is expreſſed in the firſt (tranſverſe) line, where the numbers are all *Monadicks*, or units.

2. The ſame happens, if we would *take all*, (or *leave none*.) For here alſo there can be no variety of choice, whatever be the number of things expoſed, *a*, *b*, *c*, &c. And this, in the table, we expreſs in the firſt (erect) column, where alſo the numbers are all *Monadicks*.

3. If

3. If we would *take One*, it is manifeſt, that there are as many caſes or varieties of choice, as is the number of things. For that One may be any one of them, as *a*, *b*, *c*, *d*, *e*, *f*, *g*; which is expreſſed in the ſecond line, where the numbers are in their natural order or conſecution, 1, 2, 3, &c, which I call *Laterals*.

4. The ſame happens, if, taking all the reſt, we *leave One*; that is, if we take *All but One*. For it is manifeſt, there is the ſame variety of *leaving One* as of *taking One*, as *abcdef*, *abcdeg*, *abcdfg*, *abcefg*, *abdefg*, *acdefg*, *bcdefg*, which is ſignified in the ſecond column, where the numbers are alſo *Laterals*.

5. If we would take *Two*; It is manifeſt, that we may firſt take *a*, combined with any other of the reſt; as *ab*, *ac*, *ad*, *ae*, *af*, *ag*; the number of which combinations are therefore as many as the number of things wanting one. We may then take *b* (omitting its combination with *a*, as being already taken,) combined with every of thoſe which follow it; as *bc*, *bd*, *be*, *bf*, *bg*; the number of which combinations are therefore as many as the number of things expoſed, *wanting Two*. In like manner, *c*, (omitting its combinations with *a* and *b*, becauſe *ca*, *cb*, are but the ſame with *ac*, *bc*, already taken,) may be further combined with every of thoſe which follow it, (which are ſo many as is the number of things expoſed *wanting Three*,) as *cd*, *ce*, *cf*, *cg*; and the fourth *d*, (omitting *da*, *db*, *dc*, as being the ſame with *ad*, *bd*, *cd*, already taken,) may be further combined with every of thoſe which follow, (which are as many as the number of things *wanting Four*,) as *de*, *df*, *dg*. And in like manner for the fifth, ſixth, &c; each of which affords new combinations fewer by one than that next before it, till at length we come to 1, as *ef*, *eg*, and *fg*. So that the number of all theſe combinations, is the aggregate of all the numbers in the ſame line ſo far; that is, in the preſent caſe, (the number

ab	*ac*	*ad*	*ae*	*af*	*ag*	6
	bc	*bd*	*be*	*bf*	*bg*	5
		cd	*ce*	*cf*	*cg*	4
			de	*df*	*dg*	3
				ef	*eg*	2
					fg	1
						21

ber of things expoſed being 7,) the combinations are, $6 + 5 + 4 + 3 + 2 + 1 = 21$. To which anſwers (In the third tranſverſe, or horizontal line of the foregoing Table of the Figurate numbers,) the *Triangular* number 21, juſt under the number 6, (which is leſs by one than the number of things expoſed.) Such *Triangular* numbers, being the aggregate of all the *Laterals* ſo far. And univerſally, (whatever be the number of things expoſed) the number of Two's, is *a Triangular number, whoſe ſide is leſs by one than the number of things expoſed.*

6. The ſame happens, if we are to take *All but Two*; for there is the ſame variety of leaving Two, as of taking Two; that is, in both caſes, ſo many as is the triangular number, whoſe ſide is leſs by one than the number of things expoſed, which (in the table,) is ſignified in the third column, whoſe numbers are the ſame with thoſe of the third line,

7. If we would take *Three*, it is manifeſt, that firſt, *ab*, (the firſt and ſecond,) may be combined with every of thoſe that follow; the number of which are as many as the things expoſed *wanting Two*, (which therefore afford us ſo many different Triads, or Three's,) as *abc*, *abd*, *abe*, *abf*, *abg*. Then that *ac* (the firſt and third,) may be combined (omitting *acb*, as being the ſame with *abc* already taken,) with every of thoſe that follow, (which therefore afford us ſo many new *Three's*, as is the number of things *wanting Three*,) as *acd*, *ace*, *acf*, *acg*. And, in like manner, *a* coupled with thoſe that follow, (as *ad*, *ae*, *af*,) may each of them be further combined with their reſpective ſubſequents, affording each of them new Triads, fewer by one than that next before it, till at length we come to 1, as *ade*, *adf*, *adg*, and *aef*, *aeg*, and *afg*. (But *ag* affords none, becauſe *g* being the laſt, there is none remaining with which it might be combined.) The aggregate of all which, is a Triangular number (as being an aggregate of Laterals,) whoſe ſide is leſs by two, than the number of things expoſed; that is, in the preſent

abc abd abe abf abg	5
acd ace acf acg	4
ade adf adg	3
aef aeg	2
afg	1
	15

prefent cafe, 5 + 4 + 3 + 2 + 1 = 15; which is a triangular number of the fide 5, which is lefs by two, than 7, the number of things expofed, in all which, *a* is one of the Ingredients.

In like manner (omitting all the Triads wherein *a* is an Ingredient, as being already taken,) *bc* (the fecond and third) may be further combined with each of thofe that follow *d, e, f, g,* affording us as many new Triads as did *ac*, (which was before fo compounded,) that is, fo many as is the number of things *wanting Three.* And then again *bd, be, bf,* afford as many as *ad, ae, af,* did before. Which afford us a new Triangular number, whofe fide is lefs by one than that we had before; that is, 4 + 3 + 2 + 1 = 10, whofe fide is 4; in all which Triads *b* is the leader.

bcd	*bce*	*bcf*	*bcg*	4
	bde	*bdf*	*bdg*	3
		bef	*beg*	2
			bfg	1
				10

In the fame manner may be fhewed, that (omitting the combinations of *a* and *b*,) thofe Triads wherein *c* is the leader, will give another Triangular number, whofe fide is yet lefs by one, and fo onward continually till we come at 1: as 3 + 2 + 1 = 6, a triangular number whofe fide is 3; and 2 + 1 = 3, a Triangular number, whofe fide is 2; and 1, a Triangular number whofe fide is alfo 1.

cde	*cdf*	*cdg*	3
	cef	*ceg*	2
		cfg	1
			6

def	*deg*	2
	dfg	1
		3

efg	1
	1

And then the aggregate of thefe Triangulars is 35, a *Pyramidal* number, which (in the fourth line,) ftands next under 15, the greateft of them, whofe fide is lefs by two, than the number of things expofed; that is, a *Pyramidal number whofe fide is lefs by two than the number of things expofed.* And fo many are the feveral Triads which may be had in fuch number of things expofed; that is, in the prefent cafe, 15 +

5, 4, 3, 2, 1	15
4, 3, 2, 1	10
3, 2, 1	6
2, 1	3
1	1
	35

 10 +

10 + 6 + 3 + 1 = 35, which is repreſented in the fourth line, which is of pyramidal numbers.

8. The ſame happens, if inſtead of taking Three, we take *All but Three.* For the ſame variety of caſes happens, if now we take what were before left, and leave what were then taken. And as that is repreſented in the fourth line, ſo this is in the fourth column.

9. If we would take *Four*; then, with *a*, may be made ſo many Fours (or Quaternions,) as may be formed Triads of thoſe that follow, (as *b*, *c*, *d*, *e*, *f*, *g*,) that is, (by art. 7,) a pyramidal number whoſe ſide is leſs by Two than the number of theſe; that is, leſs by Three than the number of things expoſed; that is, in the preſent caſe, 20; which is a Pyramidal number of the ſide 4, which is leſs by Three, than 7.

In like manner, (omitting *a*,) there may with *b*, be ſo many Quaternions formed, as may be Triads of thoſe that follow it, (as *c*, *d*, *e*, *f*, *g*;) that is, a Pyramidal number whoſe ſide is leſs by 1, than that foregoing; that is, 10; whoſe ſide 3 is leſs by 4 than 7.

And (omitting *a*, *b*,) there may with *c* be formed ſo many Quaternions, as may of thoſe that follow it (*d*, *e*, *f*, *g*,) be formed Triads; that is, a Pyramidal number whoſe ſide is yet leſs by 1: that is, 4, whoſe ſide is 2. And ſo onward, till we come at 1.

And then the aggregate of all theſe Pyramidals, that is, the number in the fifth line, next under the greateſt of
20 them, is (what they call) a *Trianguli-triangular* number,
10 whoſe ſide is leſs by three than the number of things
4 expoſed. That is, in the preſent caſe, (where the
1 number of things is 7,) 20 + 10 + 4 + 1 = 35, (a
35 trianguli-triangular number, of the ſide 4 = 7 — 3,) is the number of different Quaternions which may be had when the things expoſed are 7.

(If any like not the name of *trianguli-triangular*, and ſo of the reſt that follow; I am content to change them. For I am not fond of them, but uſe them becauſe I find them.)

Which

Which number is the ſame which before we had for Three's; which hence comes to paſs, becauſe, when the number of things is 7, the number 4, is the ſame with *All wanting* 3; where the variety is the ſame as if 3 were taken; as is ſhewed in the preceeding article.

10. The ſame happens, (for the reaſons already ſhewed,) if we were to take *All wanting Four*. And as that is to be found in the fifth line, ſo this, in the fifth column, whoſe numbers are the ſame with thoſe of the fifth line.

11. In the ſame manner will be ſhewed, that, if we would take *Five* (or *All but Five*,) the varieties are then ſo many as is the aggregate of the numbers in the fifth line, end-
ing with that whoſe ſide is *leſs by Four* than the number 15
of things expoſed. That is, the number in the ſixth 5
line (which is *Trianguli-pyramidals*) next under the 1
greateſt of thoſe, whoſe ſide is *leſs by Four* than the 21
number of things expoſed. That is, in the preſent
caſe, 15 + 5 + 1 = 21, a *Trianguli-pyramidal* number, whoſe ſide is 3 = 7 — 4. And ſo, if *Six* are to be taken, (or *All but Six*,) the varieties are ſo many, as is the aggregate of numbers in the ſixth line (or the number anſwer-
ing thereunto in the ſeventh,) ending with that whoſe 6
ſide is *leſs by Five* than the number of things expoſed. 1
And ſo for *Seven*, *Eight*, &c, (or all but ſeven, eight, 7
&c,) we are to take the numbers of the following lines, ending with that whoſe ſide is leſs by one, than that for the line next above. As, in the preſent, (where 7 is the number of things expoſed,) the number of ſixes is 7; the number of ſevens is 1.

12. All theſe varieties of choice, for any number of things expoſed, are found in the Table foregoing, in a rank of numbers obliquely deſcending; in which that number which is the number of things expoſed, is to be found in the ſecond line, and again in the ſecond column, both which are of Laterals. As, in the preſent caſe (where 7 is the number of things expoſed,) in the oblique deſcent paſſing

by

by 7 in the ſecond line, and again in the ſecond column; we have the numbers 1, 7, 21, 35, 35, 21, 7, 1; which repreſent the variety of caſes for taking, 0, 1, 2, 3, 4, 5, 6, 7. And the like for any other number of things expoſed.

13. And theſe numbers (as appears upon view,) are the ſame with thoſe which are called *unciæ*, prefixed to the proportionals that conſtitute the reſpective powers of a binomial root; or, (which is the ſame) the reſpective powers of 1 + 1 conſidered as a binomial root. That is, the root, ſquare, cube, fourth, fifth power, &c, of 1 + 1, according as the number of things expoſed are 1, 2, 3, 4, 5, 6, &c.

14. The table thus begun, is eaſily continued as far as there is occaſion: for the number of each place, is the aggregate of two numbers, whereof one is next above it, and the other next before it, as 15 = 5 + 10, 20 = 10 + 10, 35 = 20 + 15. And ſo every where.

15. Having therefore any number of things expoſed, let that number be ſought in the ſecond line, (which is of Laterals,) and again in the ſecond column; and then, in the ſloping rank of numbers paſſing through theſe two, we have the number of caſes for taking 0, 1, 2, 3, 4, &c, in ſuch order as the index on the ſide directs; and likewiſe for taking *All but* 0, 1, 2, 3, 4, &c, in ſuch order as the index on the top directs.

16. And if we would have the ſum of all theſe varieties (for any ſuch number of things propoſed) all together, it is had by adding the numbers of ſuch ſloping rank; as in the preſent caſe, 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128.

17. Which number is always that power of the number 2, (that is, of 1 + 1) which is of ſo many dimenſions as is the number of things expoſed, (or that power whoſe exponent

is

is ſuch number,) that is, the product of ſo many two's continually multiplied, (as, in the preſent caſe, $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$;) or, 1 ſo many times doubled as is the number of things expoſed. That is, for 0, it is 1. (For here, to take all, or to leave all, is but one and the ſame caſe.) For 1, it is (the ſide) $1 + 1$. For 2, (the ſquare) $1 + 2 + 1 = 4$. For 3, (the cube) $1 + 3 + 3 + 1 = 8$. For 4, (the biquadrate) $1 + 4 + 6 + 4 + 1 = 16$. And ſo of the reſt.

18. And thus far we have conſidered the variety of caſes concerning taking or leaving, *None*, *One*, *Two*, *Three*, &c, of any number of things expoſed, without regarding the order of them, ſo that *abc*, *acb*, *bac*, *bca*, &c, are reputed for one and the ſame caſe. But if the different alternations, or changes of order, in the ſame things, be accounted as different caſes; this we are to conſider in the next Chapter. And if therein, ſome two or more are indifferently reputed as one and the ſame, or indifferently to be taken each for other; what abatement of the former number will hereupon ariſe, is conſidered in the ſame Chapter.

19. If, by *Combination*, we underſtand the taking of two or more, (but not of one, or none;) then, out of the number of caſes before found, we muſt abate ſo many as is the number of things expoſed, and one more. For, of thoſe, ſo many as is the number of things expoſed, anſwers to the caſes of 1. And one more, anſwers to the caſe of taking *None*. But all the reſt are combinations in that ſenſe. For though *Combination* (as coming from *Bini*,) in its proper ſignification extend only to the taking of *couples*, (or Two's;) yet in common acceptation the word is now uſed of greater numbers. And, in Engliſh alſo, we ſcruple not to ſay, that Three, or Four, (or more than ſo,) are *coupled* together, that is, connected.

20. If, out of the former number of caſes, we pleaſe to exclude that of *taking None*, or 0, (becauſe, *to take none*, is *not to take*,) then is the number of caſes fewer by one, than

than is above expreſſed. And ſo we have the caſes of taking one or more. And ſo many are the number of *Diviſors* of a number compounded of ſo many different Prime numbers continually multiplied, as are the caſes of taking one or more of ſo many things expoſed.

21. And if further we abate one more (which anſwers to the caſe of taking all;) then have we the number of *Aliquot Parts* of a number ſo compoſed of different Primes or Incompoſite numbers. The number of *Aliquot Parts* being fewer by one, than is the number of diviſors.

I ſhall ſubjoin to this Chapter (as properly appertaining to this place,) an Explication of the *Rule of Combination*, which I find in *Buckley*'s Arithmetick, at the end of *Seton*'s Logick, (in the Cambridge edition;) which (becauſe obſcure,) Mr. George Fairfax (a Teacher of the Mathematicks then in Oxford,) deſired me to explain; to whom (Sept. 12, 1674,) I gave the explication under written; Conſonant to the doctrine of this Treatiſe, (which had been long before written, and was the ſubject of divers public Lectures in Oxford, in the years 1671, 1672.)

Regula Combinationis.

Quot fuerint Numeri, quos Combinare velimus;
Tot ſint et ſeries, quibus eſt proportio dupla;
Quarum principium ducatur ſemper ab Uno.
Omnes has ſeries conjunge per Additionem.
Producto, numerum quot Combinatio conſtat,
Aufer. Quod ſupereſt, numerum citat; unde patebit,
Quot faciant numeros diſtinctos, undique ſiquis
Propoſitos numeros velit in ſe Multiplicare.
Si nihil à ſummâ prædictâ ſurripiatur;
Reſtabunt partes Aliquotæ, quæ numerabunt
Illum, qui numeros eſt inter Maximus omnes,
Ex ductu in ſeſe numerorum provenientem.

I have taken the liberty, to alter the pointing (ſo as to make the ſenſe the clearer,) and to reſtore (in the ſecond verſe)

verse) *sint*, for *sunt*; and (in the third verse) *principium*, for *principio*; which had been misprinted. And (in the fifth verse) I have restored *numerum*, for *numeros*; for it is but one Number that is to be subducted, namely, the Number of those Numbers which are to be Combined. My Explication was this:

" Let as many Numbers as you please, be proposed to be " Combined: Suppose *Five*, which we will call *a b c d e*.

1	*a*	*ab*	*abc*	*abcd*	*abcde*	
2	*b*	*ac*	*abd*	*abce*	—	
4	*c*	*ad*	*abe*	*abde*	1	
8	*d*	*ae*	*acd*	*acde*		
16	*e*	*bc*	*ace*	*bcde*		
—	—	*bd*	*ade*	—	5	
31	5	*be*	*bcd*	5	10	10
-5		*cd*	*bce*		10	10
—		*ce*	*bde*		5	5
26		*de*	*cde*		1	1
		—	—		—	—
		10	10		31	26

" Put, in so many Lines, Numbers, in duple proportion, " beginning with 1.

" The Sum (31) is the Number of Sumptions, or Elections; " wherein, one or more of them, may several ways be " taken.

" Hence subduct (5) the Number of the Numbers pro- " posed; because each of them may once be taken singly.

" And the Remainder (26) shews how many ways they " may be taken in Combination; (namely, Two or more at " once.)

" And, consequently, how many Products may be had " by the Multiplication of any two or more of them so " taken.

" But the same Sum (31) without such Subduction, shews " how many Aliquot Parts there are in the greatest of those " Products, (that is, in the Number made by the continual " Multiplication of all the Numbers proposed,) *a b c d e*. " For every one of those Sumptions, are Aliquot Parts of

" *a b c d e*, except the laſt, (which is the whole,) and inſtead
" thereof, 1 is alſo an Aliquot Part; which makes the
" number of Aliquot Parts, the ſame with the Number of
" Sumptions.

" Only here is to be underſtood, (which the Rule ſhould
" have intimated;) that, all the Numbers propoſed, are to
" be Prime Numbers, and each diſtinct from the other. For
" if any of them be Compound Numbers, or any Two of
" them be the ſame, the Rule for Aliquot Parts will not
" hold."

CHAP. II.

Of Alternations, or the different Change of Order, in any Number of Things propoſed.

SUPPOSE we a certain Number of things expoſed, different each from other, as *a*, *b*, *c*, *d*, *e*, &c. The queſtion is, how many ways the order of theſe may be varied? as, for inſtance, how many changes may be Rung upon a certain Number of Bells; or, how many ways (by way of Anagram) a certain Number of (different) Letters may be differently ordered?

$a \} 1$

$\left.\begin{matrix} ab \\ ba \end{matrix}\right\} 2$

$1 \times 2 = 2$

1. If the thing expoſed be but *One*, as *a*, it is certain, that the order can be but one. That is 1.

2. If *Two* be expoſed, as *a*, *b*, it is alſo manifeſt, that they may be taken in a double order, as *ab*, *ba*, and no more. That is $1 \times 2 = 2$.

3. If

3. If *Three* be expoſed; as *a*, *b*, *c*: Then, beginning with *a*, the other two *b*, *c*, may (by art. 2,) be diſpoſed according to Two different orders, as *bc*, *cb*; whence ariſe Two Changes (or varieties of order) beginning with *a*, as *abc*, *acb*: And, in like manner it may be ſhewed, that there be as many beginning with *b*; becauſe the other two, *a*, *c*, may be ſo varied, as *bac*, *bca*. And again as many beginning with *c*, as *cab*, *cba*. And therefore, in all, Three times Two. That is, $1 \times 2, \times 3 = 6$.

abc *acb*	2
bca *bac*	2
cab *cba*	2
$2 \times 3 = 6$	

4. If *Four* be expoſed, as *a*, *b*, *c*, *d*; Then, beginning with *a*, the other Three may (by art. preceeding) be diſpoſed ſix ſeveral ways. And (by the ſame reaſon) as many beginning with *b*, and as many beginning with *c*, and as many beginning with *d*. And therefore, in all, Four times ſix, or 24. That is, the Number anſwering to the caſe next foregoing, ſo many times taken as is the Number of things here expoſed. That is, $1 \times 2 \times 3, \times 4 = 6 \times 4 = 24$.

5. And in like manner it may be ſhewed, that this Number 24 Multiplied by 5, that is $120 = 24 \times 5 = 1 \times 2 \times 3 \times 4 \times 5$, is the number of alternations (or changes of order) of *Five* things expoſed. (Or, the Number of Changes on Five Bells.) For each of theſe five being put in the firſt place, the other four will (by art. preceeding) admit of 24 varieties, that is, in all, five times 24. And, in like manner, this Number 120 Multiplied by 6, ſhews the Number of Alternations of 6 things expoſed; and ſo onward, by continual Multiplication by the conſequent Numbers 7, 8, 9, &c.

abcd *abdc* *acbd* *acdb* *adbc* *adcb*	6
bacd *badc* *bcad* *bcda* *bdac* *bdca*	6
cabd *cadb* *cbad* *cbda* *cdab* *cdba*	6
dabc *dacb* *dbac* *dbca* *dcab* *dcba*	6
$4 \times 6 = 24$	

6. That is, how many ſo ever of Numbers, in their natural Conſecution, beginning from 1, being continually Mul-

tiplied, give us the Number of Alternations (or Change of order) of which ſo many things are capable as is the laſt of the Numbers ſo Multiplied. As for inſtance, the Number of Changes in Ringing Five Bells, is $1 \times 2 \times 3 \times 4 \times 5 = 120$. In Six Bells, $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 120 \times 6 = 720$. In Seven Bells, $720 \times 7 = 5040$. In Eight Bells, $5040 \times 8 = 40320$. And ſo onward, as far as we pleaſe.

Thus *Voſſius* tells us, (*Cap.* 7, *De Scientiis Mathematicis*,) That if an Hoſt promiſe to entertain ſeven Gueſts ſo long as they ſit every day in a different order, this extends to 14 years. He means, *almoſt* ſo many years, namely, 5040 days, which of 14 years wants 73 or 74 days, according as the Leap-years may chance to fall.

7. This Number of Alternations, according as the Number of things expoſed doth increaſe, will proceed to a vaſt Multitude beyond what at firſt one would expect. As for Example, the 24 Letters will admit of ſo many Varieties or Alternations in Changing their order, as that if ſo many Bells were to be Rung according to all thoſe Changes, it could not have been diſpatched (as the Learned *John Gerard Voſſius*, in the place laſt cited, doth obſerve,) from the beginning of the World to this day. I add; no, nor if for every Minute of an hour which hath paſſed, there had paſſed *Ten Thouſand Thouſand Years*; as will appear by the following Computation.

1	1 ×
2	2 ×
6	3 ×
24	4 ×
120	5 ×
720	6 ×
5,040	7 ×
40,320	8 ×
362,880	9 ×
3,628,800	10 ×
39,916,800	11 ×
479,001,600	12 ×
6,227,020,800	13 ×
87,178,291,200	14 ×
1,307,674,368,000	15 ×
20,922,789,888,000	16 ×
355,687,428,096,000	17 ×
6,402,373,705,728,000	18 ×
121,645,100,408,832,000	19 ×
2,432,902,008,176,640,000	20 ×
51,090,942,171,709,440,000	21 ×
1,124,000,727,777,607,680,000	22 ×
25,852,016,738,884,976,640,000	23 ×
620,448,401,733,239,439,360,000	24 ×

In 1 year.
365¼ days.
× 24
1460
730
6
8766 hours.
× 60
525,960 Minutes
In 6000 years.
3,155,760,000 Minutes
× 5
15,778,800,000 Changes.
525,960 Min. in 1 year.
946728000000
1420092
788940
315576
788940
8,299,017,648,000,000
10,000,000
82,990,176,480,000,000,000,000

For, ſuppoſing in one year, 365¼ days; and, from the beginning of the World, to have paſſed 6000 years; (both of which ſuppoſitions are at the largeſt,) and therefore the Number of minutes in all that time, 3,155,760,000. Suppoſe we then, in every Minute of an hour, 5 Changes to be diſpatched, that is, (becauſe of 24 Bells) 120 ſtrokes ſucceſſively one after another, (which allowance is alſo at the largeſt;) and therefore, in 6000 years, 15,778,800,000 Changes, which Number if we Multiply by 525,960, (the Number

ber

ber of Minutes in one year,) we have 8,299,017,648,000,000 for the Number of Changes to be dispatched in so many years as there have been Minutes, which Multiplied by 10,000,000, (Ten Thousand Thousand, or Ten Millions,) will be but 82,990,176,480,000,000,000,000, which is less than 620,448,401,733,239,439,360,000, the Number of Changes whereof 24 Bells are capable.

Nay, if we should proceed no further than to 14 Bells, and allow 10 Changes (that is, 140 strokes) to every Minute, the Number of Minutes requisite to Ring them all would be 8,717,829,120, (a tenth part of the Number of Changes,) which is more than double (almost treble) the Number of Minutes in 6000 Years; and would require more than 16 Thousand Years (yea, more than 16,575 Years) to Ring them all.

8. Hence it may appear, how many ways the Letters of a Name or Word, (supposing them to be all several,) may be differently disposed by way of Anagram, (out of which those that are of use may be selected, neglecting the rest,) by art. 6. For Example, the Word *R O M A*, (consisting of four different Letters) may admit of Changes $24 = 1 \times 2 \times 3 \times 4$.

Roma	orma	mroa	arom
roam	*oram*	mrao	*armo*
rmoa	omra	*mora*	aorm
rmao	omar	moar	aomr
raom	oarm	*maro*	amro
ramo	oamr	maor	*amor*

Of which (in Latin) these seven are only useful; *Roma*, *ramo*, *oram*, *mora*, *maro*, *armo*, *amor*. The other forms are useless, as affording no Latin Word of known signification.

9. But in case some one or more of the Letters do occur more than once, the Number of Alternations so found as before, must be divided by such Number or Numbers as such repetitions do require: Namely, if the same Letter do twice occur, we are to divide by 2; if three times, by 6; if four times,

times, by 24; and ſo onward, according to the varieties that ſuch a Number is capable of. For, if the Letters *a* and *b* be reputed for the ſame; then, whereas (the reſt remaining as before) *ab* and *ba* would every where afford two varieties, they are in this caſe to paſs for one, and therefore the Number of caſes will be only half ſo many as otherwiſe they would be. In like manner (the reſt remaining as before) *abc* would every where (according as they may change places one with another) afford ſix varieties; but in caſe the three Letters *a*, *b*, *c* be conſidered as being all the ſame, or as being *a*, *a*, *a*, theſe Six caſes muſt then paſs but for one. And in like manner, if *abcd* be ſeverals, they afford (the reſt remaining as before) 24 varieties; but, if the ſame, theſe 24 muſt paſs but for one: And the like in other caſes. And, if more Letters be ſo repeated, there muſt be for each of them ſuch diviſion.

For Example, the Word *MESSES* having 6 Letters, if they were all different, the Alternations would be $720 = 1 \times 2 \times 3 \times 4 \times 5 \times 6$. But becauſe the Letter *e* comes twice, that Number is to be divided by 2. (For if inſtead of *ee*, we put *εη*, then *mεssηs* and *mηssεs* would be two forms, both which are now Co-incident in *messes*: And ſo every where.) Again, becauſe the Letter *s* comes three times, we are (for the like reaſon) to divide by 6. (For if thoſe three were three different, they would in every poſition of the reſt, afford 6 caſes, all which are now Co-incident in *sss*.) And therefore, (becauſe both happen,) 720 being divided by 2, and again by 6, the different varieties will be $\frac{720}{2 \times 6} = 60$.

meesss	emesss	esmsse	smeess	seesms	ssmsee
mesess	emsess	esemss	smeses	seessm	ssemes
messes	emsses	esesms	smesse	sesmes	ssemse
messse	emssse	esessm	smsees	sesmse	sseems
mseess	eemsss	essmes	smsese	sesems	sseesm
mseses	eesmss	essmse	smssee	sesesm	ssesme
msesse	eessms	essems	semess	sessme	ssesem
mssees	eesssm	essesm	semses	sessem	sssmee
mssese	esmess	esssme	semsse	ssmees	ssseme
msssee	esmses	esssem	seemss	ssmese	ssseem

Of

Of all which varieties, there is none befide *messes* itfelf, that affords an ufeful Anagram.

In like manner we may fhew, that the Letters *abbcccdddd* will admit of $\frac{1\times2\times3\times4\times5\times6\times7\times8\times9\times10=3628800}{2\times6\times24=288}=12600$ feveral varieties: And *abbccdd*, of, $\frac{1\times2\times3\times4\times5\times6\times7=5040}{2\times2\times2=8}$ $=630$: And *aaabbccc*, of $\frac{1\times2\times3\times4\times5\times6\times7\times8=40320}{6\times2\times6=72}=$ 560. And the like in other cafes however varied.

10. The converfe of this, is of like ufe, when what was confidered but as one and the fame feveral times repeated, comes afterward to be diftinguifhed. For then the Number before found, is to be fo often Multiplied as the Number of things fo diftinguifhed fhall require.

As, in the Word *messes* before mentioned, where *sss* are confidered but as one Letter thrice repeated, and *ee* as the fame twice repeated, the Number of different pofitions is 60; but if *sss* be diftinguifhed as three feverals, and *ee* as two feverals, the Number of all will be $60\times6\times2=720$.

Thus *Voffius*, Cap. 7, *De Scientiis Mathematicis*, tells us that this verfe,

Rex, lex, fol, lux, dux, fons, mons, fpes, pax, petra, Chriftus.

which (confifting of 11 Words) may be turned (abfolutely) 39,916,800 ways; and fo as to preferve the Rules of an Hexameter verfe, be turned 3,628,800 ways, he fhould rather have faid 3,265,920. That is, the 9 Monofyllables (which may promifcuoufly take each other's place) 362,880 times; and *Chriftus* is capable of 9 (not 10) different pofitions; that is, in the firft, fecond, third, fourth, fifth, fixth, feventh, eighth, (but not in the ninth, and tenth,) and in the laft place; (and *petra* confined, by the nature of the verfe, to the place next before the laft fpondee.) That is, $362,880\times9=3,265,920$ ways:

He

He ſays alſo that the verſe

Tot tibi ſunt dotes, virgo, quot ſidera cælo;

may be turned abſolutely 40320 ways; and, ſo as to preſerve the verſe 1022 ways; which is very true, (and I have been told, of ſomebody, who, in praiſe of the Virgin *Mary*, had made a Book of that verſe turned ſo many ways, which was wont to be reputed the Number of the Fixed Stars, according to the ancient Catalogue of them.) But it is true alſo, that it may be turned many more ways than ſo, and yet preſerve the verſe true: Namely, 2628, retaining the quantity of the laſt Syllables in *tibi* and *virgo* as before; and 468, Changing their quantity in *virgo tibi*. That is, in all 3096 ways. As will appear by the Scheme adjoyned, and the brief Explication, (or Demonſtration) of it: which is thus to be underſtood.

Tot, *ſunt*, *quot*, which may promiſcuouſly ſupply each other's place, are (in verſe 1, 2, 3, &c,) ſet down only in this order, and ſo paſs but for one caſe; but are capable of ſix varieties; which caſe I call $a = 6$. And the like for *dotes*, *virgo*, *cælo*; which caſe I call $b = 6$. And again, *tot tibi* may change place with *ſidera*; which caſe I call $c = 2$: And, becauſe all theſe happen in verſe 1, the varieties thereby repreſented, are $abc = 72 = 6 \times 6 \times 2$. And ſo of the reſt, as the Scheme directs.

1.	*Tot*	*tibi*	*ſunt*	*dotes*	*virgo*	*quot*	*ſidera*	*cælo.*	$abc =$	72
2.					*quot*	*virgo*			$abcd =$	144
3.				*quot*	*dotes*				$ace =$	1152
4.			*dotes*	*ſunt*	*virgo*	*quot*			$abcf =$	144
5.		*ſunt*	*dotes*	*quot*	*virgo*	*tibi*			$agh =$	180
6.			*quot*	*dotes*	*tibi*	*virgo*			$abi =$	324
7.					*virgo*	*tibi*			$abi =$	324
8.		*dotes*	*tibi*	*ſunt*	*virgo*	*quot*			$ab =$	36
9.					*quot*	*virgo*			$ak =$	108
10.			*ſunt*	*virgo*	*quot*	*tibi*			$ablm =$	144
									virgo tibi	2628

11.	*Virgo tibi tot ſunt dotes quot ſidera cælo:*	*an* =	36
12.	*quot dotes*	*an* =	36
13.	*dotes ſunt quot*	*ao* =	36
14.	*Tot ſunt virgo tibi dotes*	*an* =	36
15.	*quot dotes*	*an* =	36
16.	*dotes virgo tibi quot*	*an* =	36
17.	*Tot dotes ſunt*	*ap* =	12
18.	*ſunt dotes ſidera cælo virgo tibi quot*	*aq* =	144
19.	*dotes ſunt*	*apr* =	24
20.	*cælo ſunt ſidera*	*ap* =	12
21.	*ſidera tot dotes ſunt cælo*	*apr* =	24
22.	*cælo ſunt*	*ap* =	12
23.	*dotes tot*	*apr* =	24
		virgo tibi	468
		tibi virgo	2628
			3096

Tot, ſunt, quot, $a = 6$. *dotes, virgo, cælo,* $b = 6$.
tot tibi, ſidera, $c = 2$. *tot tibi, virgo,* $d = 2$.

Tot tibi, ſunt quot, dotes, virgo, cælo; $e = 120 - 24 = 120 \times \frac{4}{5} = 96$.

(Becauſe *tot tibi* cannot ſupply the place of *cælo,* as of the reſt.) *Tot tibi, dotes,* $f = 2$. *Tot ſunt, dotes, virgo, cælo,* $g = 24$. *Quot tibi, ſidera,* $k = 1\frac{1}{4}$.

(Becauſe when *tot ſunt,* or its Equivalent *ſunt quot,* comes next before *tibi,* which is a fourth part of the caſes contained in *g,* then will *quot tibi,* change with *ſidera;* which adds $\frac{1}{4}$ of what was before.) *Tot ſunt,* (and *ſunt quot,*) *dotes, virgo, cælo,* $i = 9$.

(Becauſe *dotes, virgo, cælo,* contained in *b,* may each of them change with *tot ſunt,* which Multiplies by 4, or adds a Triple to what was before, as at *g,* and $\frac{1}{3}$ of that Triple, or $\frac{1}{4}$ of that Quadruple, as at *h;* that is, it adds a Quadruple or Multiplies by 5: And again, each of them with *ſunt quot,* which, for the ſame reaſon, adds another Quadruple: Therefore both together, add an Octuple, or Multiply by 9.)

Dotes,

Dotes, sunt quot, virgo, cælo, $k = 24 - 6 = 24 \times \frac{3}{4} = 18$.

(Because, if *sunt quot* supply the place of *dotes*, it will be Co-incident with some of the cases of ver. 3.)

Quot tibi, sidera, $e = 2$.
tot sunt, dotes, cælo, $n = 6$.
dotes, cælo, $p = 2$.
sidera, cælo, $r = 2$.

virgo, quot tibi, $m = 2$.
dotes, sunt quot, cælo, $o = 6$.
tot sunt, dotes, sidera, cælo, $q = 24$.

I will not be positive that there may not be some other Changes: (and then, those may be added to these :) Or, that most of these be twice repeated, (and if so, those are to be abated out of the number :) But I do not, at present, discern either the one and other *.

CHAP. III.

Of the Divisors, and Aliquot Parts, of a Number proposed.

1. BY *Number*, I here understand only Integer Numbers, as 1, 2, 3, 4, 5, &c. Not Fractions, as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, &c. Or Mixed, as $1\frac{1}{2}$, $2\frac{1}{3}$, $3\frac{2}{5}$, &c. Much less Surds, as $\sqrt{2}$, $\sqrt{5}$, $\sqrt{6}$, &c.

2. By the *Divisor* of a Number, I here understand, such Integer as doth measure such Number; that is, being once or oftener taken doth equal it. So, of the Number 6, the Divisors are, 1, 2, 3, 6: Because 6, once taken; and 3, twice taken; and 2, thrice; and 1, six times taken; do equal 6.

$$1)6(6;\ 2)6(3;\ 3)6(2;\ 6)6(1.\quad 6 = 1 \times 6 = 2 \times 3 = 3 \times 2 = 6 \times 1.$$

* The number of all the possible variations of the words in this Hexameter Verse, without destroying the measure of it, has been investigated with greater accuracy by Mr. *James Bernouilli*, in the second part of his excellent Treatise, *De Arte Conjectandi*, and is there found to be 3312. See above, pages 8, 9, and 10.

3. By *Aliquot Part* of a Number, I underſtand ſuch a Diviſor as is leſs than it. As of 6, the Aliquot Parts are 1, 2, 3; but not 6. For, though 6 be alſo a Diviſor of itſelf; yet not an Aliquot Part; becauſe the Word *Part* implies ſomewhat leſs than the whole.

4. The Number of Aliquot Parts, therefore, is always leſs by one than the Number of Diviſors. Becauſe all the Diviſors except one, are Aliquot Parts; all the Aliquot Parts are Diviſors, and there is likewiſe one more Diviſor of the Number, to wit, the whole Number itſelf.

5. So that, the Number of Diviſors being given, the Number of Aliquot Parts is given alſo. And contrarywiſe; if this, then that. As, of the Number 6, the Diviſors being 4, the Aliquot Parts are 3, (that is, $4 - 1$.) And, theſe being 3, the Diviſors are $4 = 3 + 1$.

6. It is manifeſt, that the Number 1, hath no Aliquot Part, and but one Diviſor, that is 1. Becauſe there is no Number leſs than itſelf that may be a part of it: But it meaſures itſelf; and therefore is its own Diviſor.

7. Any other Prime Number hath one Aliquot Part, and Two Diviſors. For a *Prime Number*, we call, ſuch as is meaſured (beſide itſelf) by no other Number but an Unit. As 2, 3, 5, 7, 11, &c. Each of which are meaſured by 1, and by itſelf; but not by any other Number. And hath therefore 2 Diviſors, and 1 Aliquot Part; but no more.

8. Every *Power* of a *Prime Number* (other than of 1, which here is underſtood to be excluded,) hath ſo many Aliquot Parts as are the dimenſions of ſuch Power; and one Diviſor more than ſo. As (ſuppoſing a, b, c, &c, to be ſo many Prime Numbers;) a hath two diviſors (1 and a;) a^2 or aa hath three, (1, a, aa;) a^3, or aaa, hath four, (1, a, aa, aaa;) and ſo of the reſt. That is, the Number of Diviſors is one more than the Number of Dimenſions. Becauſe 1, and all the Degrees of ſuch Power (not higher than itſelf) are Diviſors; but not any other Number, if a be a Prime. That is, one more than the Number of Dimenſions: Of which the greateſt Diviſor (being the whole) is not an Aliquot Part; and

and therefore the Aliquot Parts are juſt ſo many as are the Dimenſions. Thus of 8 (the Cube of 2) the Diviſors are four, (1, 2, 4, 8 ;) the Aliquot Parts are three, (1, 2, 4;) Of 81 (the Biquadrate of 3) the Diviſors are five, (1, 3, 9, 27, 81 ;) the Aliquot Parts are four, (1, 3, 9, 27,) juſt ſo many as are the Dimenſions. That is, (of ſuch Biquadrate) the Diviſors are 1, *a*, *aa*, *aaa*, *aaaa* ; the Aliquot Parts 1, *a*, *aa*, *aaa* ; and ſo every where : For, though the higheſt Dimenſion came not into the Number of Aliquot Parts, yet 1 being ſupernumerary, makes the Aliquot Parts juſt as many as the Dimenſions.

9. If a Prime Number, or any Power thereof, be Multiplied by any other Prime Number, or any Power hereof ; the Product hath ſo many Diviſors, as is the Number of Diviſors in That, Multiplied by the Number of Diviſors in This ; and, therefore, the Aliquot Parts fewer by one than ſo.

For Example : Let *a*, *b*, be two different Prime Numbers, (ſuppoſe 2, 3 ;) and certain Powers thereof, as a^3, b^2, (that is 8, 9,) the Product a^3b^2, (that is, $72 = 8 \times 9$.) Now for as much as the Diviſors of the former 1, *a*, *aa*, *aaa* ; (that is, 1, 2, 4, 8,) divide a^3 (that is 8 ;) not only theſe, or (which is the ſame) every of theſe Multiplied by *1* ; but alſo every of them Multiplied by *b*, and by *bb*, (that is by 3, and by 9,) will divide a^3b^2. That is, every of the Diviſors of a^3, Multiplied into every of the Diviſors of b^2 ; will divide a^3b^2.

1	1		
a	2	4.	The Diviſors of a^3 Multiplied by 1.
aa	4		
aaa	8		
b	3		
ab	6	4.	The ſame Multiplied by *b*.
aab	12		
aaab	24		
bb	9		
abb	18	4.	The ſame Multiplied by *bb*.
aabb	36		
aaabb	72		
	$12 = 4 \times 3$.		

The

The Number therefore of all; is the Number of 1, *a*, *aa*, *aaa*, (that is 4,) ſo many times taken as is the Number of 1, *b*, *bb*, (that is, 3 times;) That is, $4 \times 3 = 12$: The Number of Diviſors therefore is 12; and of Aliquot Parts, 11.

10. If a Product made by the Multiplication of different Prime Numbers, or of their Powers by one another, be further Multiplied by another Prime Number different from every of thoſe: The Number of Diviſors in this new Product, will be ſo many as is the Number of Diviſors in that firſt Product Multiplied by the Number of Diviſors in the new Multiplier.

For Example: The Number of Diviſors in the Product but now mentioned a^3b^2, is 12; (as is already ſhewed:) if therefore this be Multiplied by any other Prime Number, as *c* (ſuppoſe 5,) different from *a*, *b*; (whoſe Diviſors therefore are two, 1 and *c*;) the Diviſors of the Product a^3b^2c (that is, of $72 \times 5 = 360$) will be $12 \times 2 = 24$. Namely, all thoſe (before found) which divided a^3b^2, will alſo divide a^3b^2c; or (which is the ſame) all thoſe Multiplied by 1 (which is one of the Diviſors of *c*;) and the ſame alſo Multiplied by *c*, (which are as many more;) and therefore both together are twice as many; that is, $12 \times 2 = 24$. Namely, 1, *a*, *aa*, *aaa*; *b*, *ab*, *aab*, *aaab*; *bb*, *abb*, *aabb*, *aaabb*; *c*, *ac*, *aac*, *aaac*; *bc*, *abc*, *aabc*, *aaabc*; *bbc*, *abbc*, *cabbc*, *aaabbc*. That is, 1, 2, 4, 8; 3, 6, 12, 24; 9, 18, 36, 72: 5, 10, 20, 40; 15, 30, 60, 120; 45, 90, 180, 360.

And if, for the new Multiplier $c = 5$, where taken $cc = 25$, or $ccc = 125$; (the Number of whoſe Diviſors are 3 or 4;) the Number of Diviſors of the Product $a^3b^2c^2$, or $a^3b^2c^3$, would (accordingly) be $12 \times 3 = 36$, or $12 \times 4 = 48$. (And, in like manner, for any other Power of *c*.) For now not only the Diviſors of a^3b^2 Multiplied, by 1, and by *c*; but the ſame alſo Multiplied by *cc*, (which is a third time ſo many,) will be Diviſors of $a^3b^2c^2$; and the ſame Multiplied by *ccc*, (which is as many a fourth time,) will be Diviſors of $a^3b^2c^3$.

1	1		*c*	5		*cc*	25		*ccc*	125	
a	2	4	*ac*	10	4	*acc*	50	4	*accc*	250	4
aa	4		*aac*	20		*aacc*	100		*aaccc*	500	
aaa	8		*aaac*	40		*aaacc*	200		*aaaccc*	1000	
b	3		*bc*	15		*bcc*	75		*bccc*	375	
ab	6	4	*abc*	30	4	*abcc*	150	4	*abccc*	750	4
aab	12		*aabc*	60		*aabcc*	300		*aabccc*	1500	
aaab	24		*aaabc*	120		*aaabcc*	600		*aaabccc*	3000	
bb	9		*bbc*	45		*bbcc*	225		*bbccc*	1125	
abb	18	4	*abbc*	90	4	*abbcc*	450	4	*abbccc*	2250	4
aabb	36		*aabbc*	180		*aabbcc*	900		*aabbccc*	4500	
aaabb	72		*aaabbc*	360		*aaabbcc*	1800		*aaabbccc*	9000	
		$4\times3=12$			12			12			12

$12\times4=48$

The ſame will in like manner be ſhewed, if this new Product a^3b^2c, (whoſe Diviſors are 24,) be further Multiplied by d, or dd, &c. Namely, the Diviſors of a^3b^2cd will be $24\times2=48$; and, of $a^3b^2cd^2$, $24\times3=72$. And ſo forward.

Or (which comes to the ſame paſs) if a^3b^2 (whoſe Diviſors are $12=4\times3$,) be Multiplied by cd, (whoſe Diviſors are $4=2\times2$,) or by cdd; (whoſe Diviſors are $2\times3=6$;) for then will the Diviſors of a^3b^2cd be $12\times4=48$; and of $a^3b^2cd^2$, $12\times6=72$; as before.

And in like manner, the ſame will hold, how many ſoever Prime Numbers, and what ever Powers of ſuch Primes, be ſo continually Multiplied; provided always (which is heedfully to be attended,) that ſuch Primes a, b, c, d, &c, be all different each from other.

11. If any Number however Compounded, be further multiplied by any of thoſe Primes of which it was before Compounded, or by any Power of ſuch Prime; the Number of Diviſors thence ariſing, will be ſuch as would have been by advancing that Prime ſo many Degrees higher, as is the Degree of ſuch Multiplier.

As, for inſtance, if c, d, were the ſame Prime; then inſtead of cd, whoſe Diviſors, if different, would have been

4 =

$4 = 2 \times 2$, ($1, c, d, cd$,) we are to take cc, whose Divisors are but 3, ($1, c, cc$,) because c, d, which would otherwise have been two different Divisors, are now but one and the same. And accordingly, the Divisors of a^3b^2cd, that is, (because $c = d$,) of $a^3b^2c^2$, will now be (not $12 \times 4 = 48$, as before,) but $12 \times 3 = 36$. So if a^3b^2c be Multiplied by d^2, and $d = b$. For then $a^3b^2cd^2$ is the same with a^3b^4c; and the number of Divisors (not $4 \times 3 \times 2 \times 3 = 72$, but) $4 \times 5 \times 2 = 40$. And the like in other cases, as is of itself manifest.

12. And, universally: *If a Number be made, by continual Multiplication of how many soever Prime Numbers, (different each from other,) or of any Powers of such Primes: The Number of Divisors of such Compound Number, is Compounded (by continual Multiplication) of the exponents of the Degrees of such Primes or their Powers so Compounded, increased (each of them) by* 1. *And such Number of Divisors, wanting* 1, *is the Number of Aliquot Parts.* (Which Theorem contains the main substance of the Doctrine of Aliquot Parts.

As, for the Number a^3b^2cd; the exponents of the Degrees or Dimensions of the Primes a, b, c, d, are 3, 2, 1, 1; and these increased by 1, are 4, 3, 2, 2. These, continually Multiplied, give us the Number of Divisors $4 \times 3 \times 2 \times 2 = 48$; and, of Aliquot Parts $48 - 1 = 47$. (And, in like manner, for any other Number however Compounded.) As is evident by what is before Demonstrated.

Hence we may gather the solution of the following Problems.

13. Any Number being proposed; to find how many Divisors it hath; and, how many Aliquot Parts.

Divide the Number proposed (and the Quotients arising from such Division) continually, by Prime Numbers (or the Powers of such) according as it is capable, till we come to 1. And we shall thereby find, of how many different Prime Numbers, and what Powers of them, the Number proposed is Compounded: which being done, we have the Number of Divisors

Divifors, and of Aliquot Parts, by the propofition foregoing.

As for Example; Let the Number fo propofed be 5940; we fhall find, upon Tryal, that it may be divided by 2, twice; by 3, three times; by 5, once; (by 7, not at all;) and by 11, once.

11)5)3)3)3)2)2) 5940(2970(1485(495(165(55(11(1

And may therefore be thus defigned, a^2b^3cd; where a, b, c, and d, denote the Numbers 2, 3, 5, and 11, refpectively, and the exponents of a, b, c, d, are 2, 3, 1, 1; and thefe increafed by 1, are 3, 4, 2, 2; which continually multiplied, are $3 \times 4 \times 2 \times 2 = 48$. So many therefore (by the propofition foregoing) are the Number of Divifors; and 47 the Number of Aliquot Parts.

14. Any Number being propofed; to find, what are the Divifors, and the Aliquot Parts thereof.

Firft find (as in the preceeding article) of what Prime Numbers, and what Powers of them, the Number propofed is Compounded. Then, taking any one of thofe Prime Numbers to whatever Degree it be advanced, fet down in order all the Divifors of fuch Degree. Then Multiply every of thefe by every Divifor of fuch Degree as fome other of thofe Primes is advanced to. And every of the Divifors hitherto found, by every Divifor of the Degree, to which a third Prime is advanced. And all thefe, by thofe of a fourth; and fo onward, if yet there be more Primes. (In fuch manner as is to be feen above in art. 10.) And the Number arifing from all thofe Multiplications, is the Number of the Divifors of the Number propofed: And all thefe Divifors, except itfelf, are the Aliquot Parts of it.

Thus for the Number $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 8 \times 9 \times 5$; fuppofe a^3b^2c. All the Divifors of $a^3 = 8$, are 1, a, aa, aaa; that is, 1, 2, 4, 8. Let thefe be multiplied by all the Divifors of $b^2 = 9$; which are 1, b, bb; that is, 1, 3, 9. And all the refults of thefe, by the Divifors of c; which are 1, c; that is, 1, 5. So have we all the Divifors of 360.

1	*a*	*aa*	*aaa*	1	2	4	8
b	*ab*	*aab*	*aaab*	3	6	12	24
bb	*abb*	*aabb*	*aaabb*	9	18	36	72
c	*ac*	*aac*	*aaac*	5	10	20	40
bc	*abc*	*aabc*	*aaabc*	15	30	60	120
bc	*abbc*	*aabbc*	*aaabbc*	45	90	180	360

And in like manner we may proceed, whatever Number be propofed, and howfoever Compounded.

But the fame may alfo be done in divers other methods, (for we are not confined to proceed always in the fame order,) which in the refult will be the fame with this. Provided always, in whatever order we proceed, that we be fure to take all the Prime Numbers, that are Ingredients of fuch Compound, with all the Degrees of them, and all the poffible Combinations that may be made of them, not exceeding (in any) the Number of Dimenfions which they have in the Compound. And, that we may be fure not to mifs any, it will be convenient to proceed, if not in this, at leaft in fome other regular order, that we may know when we have all. And fome other forms of procefs we may after have occafion to mention.

15. To find a Number, which fhall have juft fo many Divifors, or fo many Aliquot Parts, as is propofed: And, in how many forms the fame may be had; and, the leaft in each form; or the leaft of all, that may have fo many.

The Number of Aliquot Parts propofed, increafed by 1, is the Number of Divifors. This Number, we are to confider, how many ways it may be expreffed in Integers; whether by one alone, or by the Multiplication of two or more: (As is to be after fhewed in art. 17, 18.) And, as many ways as this may be done, fo many forms there are of Numbers which have juft fo many Divifors: Namely, for every of the Integers by which fuch Number is to be expreffed, fo many different Prime Numbers are to be affigned; and fuch Degrees or Powers thereof, whofe exponents are lefs by one than the refpective Integers which they reprefent; and thofe thoſe Powers or Degrees, (continually multiplied, if there be more of them,) will have ſuch Number of Diviſors as is required.

As for Example: If a Number be required which ſhall have 99 Aliquot Parts, or, (which is the ſame) 100 Diviſors. This Number 100, may be expreſſed by Integers (ſingle, or multiplied into one another,) nine ſeveral ways: $100 = 50 \times 2 = 25 \times 4 = 25 \times 2 \times 2 = 20 \times 5 = 10 \times 10 = 10 \times 5 \times 2 = 5 \times 5 \times 4 = 5 \times 5 \times 2 \times 2$: And ſo many ſeveral forms there are of Numbers which ſhall have 100 Diviſors, or 99 Aliquot Parts. Namely, if (for every of the forms wherein the Number 100 may be ſo deſigned) we take ſo many different Primes, as there are Integers in ſuch deſignation; and each of them advanced reſpectively to ſuch Degree whoſe exponent is leſs by one than the Integer it repreſents. As a^{99}, $a^{49}b$, $a^{24}b^3$, $a^{24}bc$, $a^{19}b^4$, a^9b^9, a^9b^4c, $a^4b^4c^3$, a^4b^4cd; whatever be thoſe Prime Numbers a, b, c, d, different each from other. (As appears from art. 12.) But not any other forms:

100	a^{99}
50×2	$a^{49}b$
25×4	$a^{24}b^3$
25×2×2	$a^{24}bc$
20×5	$a^{19}b^4$
10×10	a^9b^9
10×5×2	a^9b^4c
5×5×4	$a^4b^4c^3$
5×5×2×2	a^4b^4cd

As may be thence ſhewed, in caſe any other form be aſſigned. As, for inſtance, if any form be aſſigned wherein (whatever be the other Ingredients) there is the bare Square of a Prime Number, (ſuch as in none of theſe appears) as e^2. For whatever be the Number which the reſt of the ingredients deſign, that Number (becauſe of e^2) is to be Tripled (by art. 9.) But 100 is not the Triple of any Integer (as not being diviſible by 3:) And therefore cannot be ſo deſigned. And in like manner may be ſhewed, (with ſuch variation as the caſe ſhall require,) concerning any other form, different from thoſe aſſigned.

Now for finding the leaſt Number in each form, that ſhall have ſo many Diviſors; no more is to be done, but for a, b, c, d, &c, or ſo many of them as occur in each form reſpectively, to take ſo many of the ſmalleſt Primes, 2, 3, 5, 7, &c. And, of theſe, ſtill to aſſign the leſſer for that which is to have the greater Number of Dimenſions.

2 Q 2 (As

(As is of itfelf manifeft.) So for the form a^9b^4c, it is manifeft, that if for a, b, c, we take 2, 3, 5, the number muft needs be lefs, than if we take 2, 3, 7, or 3, 7, 11, or any other numbers: And, (fuppofing thofe three to be taken,) it muft needs be lefs if we affign $a = 2$, $b = 3$, $c = 5$, than if we affign them any otherwife. Becaufe, in the compofition, a is oftener to be repeated than b, and b than c.

Now when it appears, which is the leaft in each form; it is eafily determined upon view, which is the leaft of all. As, in the prefent cafe, putting $a = 2$, $b = 3$, $c = 5$, $d = 7$; it is eafy to judge that a^4b^4cd, that is, $16 \times 81 \times 5 \times 7 = 45360$, is the fmalleft number that can have 100 Divifors. For it is, to $a^4b^4c^3$; as $d = 7$, to $cc = 9$: And it is, to a^9b^4c; as $d = 7$, to $a^5 = 32$: And, to a^9b^9; as $cd = 35$, to $a^5b^5 = 7776$. And fo of the reft.

And, for the moft part, thofe are the fmaller numbers wherein more Primes be ingredients; than where fewer Primes, but in higher Degrees; as $ab = 2 \times 3 = 6$, is lefs than $a^3 = 8$; though each of them have four Divifors. But it is not always fo; for $a^3b = 8 \times 3 = 24$, is lefs than $abc = 2 \times 3 \times 5 = 30$; (though the number of Divifors be eight in each.) For here one Degree of a greater Prime $c = 5$, doth over-balance two Degrees of a leffer $aa = 4$.

16. It appears moreover, That, wherever the number of Divifors is odd, fuch Number is a Square: And, contrarywife, of every Square Number, the number of Divifors is odd. And, of every Non-quadrate Number, the number of Divifors is even: And, wherever the number of Divifors is even, fuch Number is a Non-quadrate Number.

For every Divifor divides the Number propofed by fome other Divifor, (whereof when one is the Divifor, the other is the Quotient;) except only the Square Root, (where the Divifor and Quotient are the fame.) All other Divifors therefore go by couples, and make an even Number: To which when the Square Root is to be added (which is the cafe

cafe of all Square Numbers, and of thefe only ;) this being folitary, makes the number of Divifors odd.

1	36	1	*aabb*
2	18	*a*	*abb*
3	12	*b*	*aab*
4	9	*aa*	*bb*
6		*ab*	

1	72	1	*aaabb*
2	36	*a*	*aabb*
3	24	*b*	*aaab*
4	18	*aa*	*abb*
6	12	*ab*	*aab*
8	9	*aaa*	*bb*

1	360	1	*aaabbc*
2	180	*a*	*aabbc*
3	120	*b*	*aaabc*
4	90	*aa*	*abbc*
5	72	*c*	*aaabb*
6	60	*ab*	*aabc*
8	45	*aaa*	*bbc*
9	40	*bb*	*aaac*
10	36	*ac*	*aabb*
12	30	*aab*	*abc*
15	24	*bc*	*aaab*
18	20	*abb*	*aac*

17. A Number being propofed ; to find, how many different ways it may be defigned by Integers ; whether fingly or by the continual Multiplication of more than one.

Firft find out (by art. 14,) what are all the Divifors of fuch propofed Number. Then, confidering them all fingly (beginning at the greateft and fo proceeding to the leffer ; that, by keeping fuch order, we may be the more fure not to mifs any ;) inquire, what Number doth with every of thefe compofe the Number propofed ; and if this chance to be a Compound, let this in like manner be refolved into its Components, (and fo onward as long as the Component is itfelf a Compound ;) whereby, having thus run through them all, we fhall meet with all the ways whereby the Number propofed may fo be defigned by Integers.

As for Example : Let fuch Number propofed, be 360 ; whofe Divifors (found by art. 14,) are 360, 180, 120, 90, 72, 60, 45, 40, 36, 30, 24, 20, 18, 15, 12, 10, 9, 8, 6, 5, 4, 3, 2, 1, where we fhall find the firft defignation to be 360, (or 360×1.) Then 180×2, 120×3, 90×4, and (becaufe 4 = 2×2,) 90×2×2. Then 72×5, 60×6; and (becaufe 6 = 3×2)

360
180×2
120×3
90×4
90×2×2
72×5
60×6
60×3×2
45×8
45×4×2
45×2×2×2
40×9
40×3×3
36×10

36×10
36×5×2
30×12
30×6×2
30×4×3
30×3×2×2
24×15
24×5×3
20×18
20×9×2
20×6×3
20×3×3×2
18×10×2
18×5×4
18×5×2×2
15×12×2
15×8×3
15×6×4
15×6×2×2
15×4×3×2
15×3×2×2×2
12×10×3
12×6×5
12×5×3×2
10×9×4
10×9×2×2
10×6×6
10×6×3×2
10×4×3×3
10× ×3×2×2
9×8×5
9×5×4×2
9×5×2×2×2
8×5×3×3
6×6×5×2
6×5×4×3
6×5×3×2×2
5×4×3×3×2
5×3×3×2×2×2

60×3×2. Then 45×8; and (becauſe 8 = 4×2 = 2×2×2.) 45×4×2, 45×2×2×2. Then 40×9; and (becauſe 9 = 3×3,) 40×3×3. Then 36×10; and (becauſe 10 = 5×2,) 36×5×2. Then 30×12; and (becauſe 12 = 6×2 = 4×3 = 3×2×2,) 30×6×2, 30×4×3, 30×3×2×2. Then 24×15, and (becauſe 15 = 5×3,) 24×5×3. Then 20×18, and (becauſe 18 = 9×2 = 6×3 = 3×3×2,) 20×9×2, 20×6×3, 20×3×3×2. Then, (omitting 18×20, as being the ſame with 20×18; and reſolving 20 = 10×2 = 5×4 = 5×2×2;) 18×10×2, 18×5×4, 18×5×2×2. Then (omitting 15×24, as being the ſame with 24×15; and ſo every where when a greater follows a leſs, as being had before; and reſolving 24 = 12×2 = 8×3 = 6×4 = 6×2×2 = 4×3×2 = 3×2×2×2;) 15×12×2, 15×8×3, 15×6×4, 15×6×2×2, 15×4×3×2, 15×3×2×2×2. In like manner (omitting ſuch Combinations of 12 as have been already,) 12×(30 = 15×2 =) 10×3, 12×6×5, 12×5×3×2. In like manner, 10×(36 = 18×2 = 12×3 =) 9×4, 10×9×2×2, 10×6×6, 10×6×3×2, 10×4×3×3, 10×3×3×2×2. Then 9×(40 = 20×2 = 10×4 =) 8×5, 9×5×4×2, 9×5×2×2×2. Then 8×(45 =) 5×3×3. Then 6×(60 =) 6×5×2, 6×5×4×3, 6×5×3×2×2. Laſtly, 5×(72 =) 4×3×3×2, 5×3×3×2×2×2. (The Diviſors 4, 3, 2, 1, afford no new caſes; becauſe every of them is leſs than 5, and cannot without it, or ſome greater Number, make up 360.) Which forms (in Number 52) are all the forms in which 360, may thus be expreſſed by by Integers. And how, to every of thefe forms, we may fit fo many forms of Numbers which fhall have 360 Divifors, is before fhewed in art. 15. As, for $5 \times 3 \times 3 \times 2 \times 2 \times 2$, $a^4b^2c^2def$: And fo of the reft.

But, why I have here omitted (for inftance) 5×72, $5\times36\times2$, $5\times24\times3$, $5\times18\times4$, $5\times18\times2\times2$, $5\times12\times6$, $5\times12\times3\times2$, $5\times9\times8$, $5\times9\times4\times2$, $5\times9\times2\times2\times2$, $5\times8\times3\times3$, $5\times6\times6\times2$, $5\times6\times4\times3$, $5\times6\times3\times2\times2$, and others of like kind; the caufe is evident: Becaufe, the Numbers 72, 36, 24, 18, 12, 9, 8, 6, being greater than 5, all the Combinations which have thefe ingredients were had before. For 5×72, is but the fame with 72×5; and fo of the reft. And it is fo ordered all along, that whenever a greater Number comes to follow a leffer, we may know that that cafe was (or fhould have been) had before.

But it is no way neceffary that we fhould always obferve this order; for the fame will hold, in whatever method we proceed: provided we be fure to take them all, in whatever order.

18. The fame alfo may be thus had, if the Number itfelf (of Divifors required) or the form thereof, be fo expreffed in Species, as it may thence appear in what form itfelf is Compounded of the ingredient Primes: As if we put a^3b^2c, for the Number $360 = 2\times2\times2\times3\times3\times5$; or for any other Number which is Compounded of the Third Degree of one Prime, Multiplied by the Second Degree of another Prime, and this by a Third Prime.

For, however we are not by this directed how to proceed (as before) from the greater to the leffer in a continual order, (becaufe the Second or Third Degree of a leffer Prime, may poffibly be greater than the firft of fome greater Prime;) yet we may thus, though in another order, meet with them all.

And it will be then convenient (beginning with 1,) to take the Species or Symbols, firft fingly, one by one, (as *a*, *b*, *c*,) in fuch order as they follow in the Alphabet. And then by Two's, (as *aa*, *ab*, *ac*, *bb*, &c,) and here, firft

thofe

thoſe that begin with *a*; and here again *aa* before *ab*, and this before *ac*, 1, &c, and then thoſe that begin with *b*; and here (omitting *ba*, as being the ſame with *ab* which was had before,) beginning with *bb*, or (in caſe there be not a ſecond *b*) with *bc*, and ſo onward: And then by Threes, and Fours, and ſo onward as there is occaſion; obſerving all along, as the caſe will permit, the Alphabetical order, (that we may be the more ſure not to miſs any.) Placing always, over againſt each, the correſpondent Diviſor; which doth, with it, conſtitute the Number propoſed. As, againſt *aa*, putting *abbc*, which, with it, compleats *aaabbc*.

1	a^3b^2c		1	*aaabbc*
a	a^2b^2c		a	*aabbc*
b	a^3bc		b	*aaabc*
c	a^3b^2		c	*aaabb*
a^2	ab^2c		*aa*	*abbc*
ab	a^2bc	Or thus rather,	*ab*	*aabc*
ac	a^2b^2		*ac*	*aabb*
b^2	a^3c		*bb*	*aaac*
bc	a^3b		*bc*	*aaab*
a^3	b^2c		*aaa*	*bbc*
a^2b	abc		*aab*	*abc*
a^2c	ab^2		*aac*	*abb*

And this we are to purſue ſo far, till, in that oppoſite rank, we meet with the ſame (in the caſe of a Square Number propoſed,) or, (if not a Square Number,) that which was next to follow, in the firſt rank. (As here, againſt *aaac*, we have *abb*; which was next to have followed if the firſt rank had proceeded.) For, when we be come ſo far, thoſe which were to have followed in the continuation of the firſt rank, do follow (in the ſame order, but going backward,) in the latter rank, till we come to the greateſt of all.

And having thus diſpoſed all the Diviſors in due order; we may then (beginning with the greateſt, and ſo proceeding backward to the leaſt,) compound each with its oppoſite, which ſtands againſt it. (As *cbbaaa*, *cbbaa* × *a*, *cbaaa* × *b*, &c.) And when that ſecond Component is itſelf a Compound, we are to reſolve it into its Components; (as *cba*×*aa*, *cba*×*a*×*a*, &c,) and ſo continually till it be reſolved into Primes.

When we have thus diſpatched all the Diviſors of the latter rank (for till then, there is no danger,) we are to take heed, that ſome of the Compoſitions already taken, be not taken a ſecond time in another order; and when they do ſo occur a ſecond time, we are to paſs them by. And accordingly, when I come at *caa*, I do not Compound this with the whole of *bba* which ſtands againſt it; (becauſe this hath been already conſidered, and there joined in all the Compoſitions that it is capable of;) but with all theſe Components of *b b a*, which had not before been fully conſidered. And when I come at *cb*: I omit, not only the whole of *baaa*, (which ſtands againſt it) but all the Components of it which have three Members, (becauſe not only thoſe of Four, but even of Three Components, have been fully diſpatched, before we come at *cb* which hath but

cbbaaa	360
cbbaa×*a*	180×2
cbaaa×*b*	120×3
bbaaa×*c*	72×5
cbba×*aa*	90×4
×*a*×*a*	90×2×2
cbaa×*ba*	60×6
×*b*×*a*	60×3×2
bbaa×*ca*	36×10
×*c*×*a*	36×5×2
caaa×*bb*	40×9
×*b*×*b*	40×3×3
baaa×*bc*	24×15
×*b*×*c*	24×5×3
cbb×*aaa*	45×8
×*aa*×*a*	45×4×2
×*a*×*a*×*a*	45×2×2×2
cba×*baa*	30×12
×*ba*×*a*	30×6×2
×*aa*×*b*	30×4×3
×*b*×*a*×*a*	30×3×2×2
bba×*caa*	18×20
×*ca*×*a*	18×10×2
×*aa*×*c*	18×4×5
×*c*×*a*×*a*	18×5×2×2
caa×*bb*×*a*	20×9×2
×*ba*×*b*	20×6×3
×*b*×*b*×*a*	20×3×3×2
baa×*cb*×*a*	12×15×2
×*ca*×*b*	12×10×3
×*ba*×*c*	12×6×5
×*c*×*b*×*a*	12×5×3×2
aaa×*cb*×*b*	8×15×3
×*bb*×*c*	8×9×5
×*c*×*b*×*b*	8×5×3×3
cb×*ba*×*aa*	15×6×4
×*a*×*a*	15×6×2×2
×*aa*×*b*×*a*	15×4×3×2
×*b*×*a*×*a*×*a*	15×3×2×2×2

bb×*ca*×*aa*	9×10×4
×*a*×*a*	9×10×2×2
×*aa*×*c*×*a*	9×4×5×2
×*c*×*a*×*a*×*a*	9×5×2×2×2
ca×*ba*×*ba*	10×6×6
×*b*×*a*	10×6×3×2
×*aa*×*b*×*b*	10×4×3×3
×*b*×*b*×*a*×*a*	10×3×3×2×2
ba×*ba*×*c*×*a*	6×6×5×2
×*aa*×*c*×*b*	6×4×5×3
×*c*×*b*×*a*×*a*	6×5×3×2×2
aa×*c*×*b*×*b*×*a*	4×5×3×3×2
c×*b*×*b*×*a*×*a*×*a*	5×3×3×2×2×2

two Components.) And when I come at *ca*, I omit *ca*×*bb*×*aa*, &c, becauſe *bb* had been before conſidered. And in like manner, at *ba*, I omit all the Compoſitions wherein *cb*, *bb*, *ca*, were ingredients; becauſe theſe had been before conſidered. And in like manner, at *aa*, and *c*, I omit all thoſe of two Members which might be Compounded with them; becauſe already had. As is to be ſeen in the order adjoined.

And over againſt the forms thus expreſſed in Species; I have ſet the Numbers anſwering to them; which are the ſame with thoſe at art. 17, but not in the ſame order. Becauſe here I was guided by the forms of Compoſition, in directing the order; but, there, by the bigneſs of the Numbers.

Having thus laid the Foundation of this Doctrine of Diviſors and Aliquot Parts; I ſhall give ſome Examples of Operations concerning them.

Examples of the foregoing Operations.

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19. Of the Number 110,880: How many are the Diviſors, and Aliquot Parts? And which be they?

The Number 110,880 divided, as is directed at art. 13, is reſolved into theſe Primes; 2, 2, 2, 2, 2, 3, 3, 5, 7, 11. And is therefore in this form a^5b^2cde.

11) 5) 7) 3) 3) 2) 2) 2) 2) 2) 110880 (55440 (27720 (13860
(6930 (3465 (1155 (385 (77 (11 (1

Or, I might at firſt cut off the Cypher; and, for it, ſet down two Diviſors 2, 5: And then, becauſe it is obvious to view, that 11,088 is diviſible by 11; I might next ſet down 11 for another Diviſor. (Becauſe by this means we come

come the fooner to fmall Numbers.) And then divide the Quotient 1008 by 2, and 3, as oft as I can; which done, we fhall have 7 for the laft Divifor. Or, I might have divided 11,088 by 9; (and for it fet down two Divifors 3, 3:) For it is obvious alfo to view that it may be fo divided; becaufe the Figures put together without regard had to the places, (as is ufual in the proofs of Multiplication and Divifion,) may be fo divided; or, cafting away 9 as oft as may be, nothing remains; or, I may fo do, for the fame reafon, with 1008; or, take any the like advantage for expedition, as the view fhall direct. For it matters not, in what order we find the Component Primes, fo we have them all.

The Number therefore appearing in this form a^5b^2cde; it is manifeft (by art. 12,) that the Number of Divifors is $6\times3\times2\times2\times2 = 144$; and, of Aliquot Parts, $144 - 1 = 143$. And thofe, (according to the method of art. 18,) are found to be thefe that follow.

1	1	110880	*aaaaabbcde*
a	2	55440	*aaaabbcde*
b	3	36960	*aaaaabcde*
c	5	22176	*aaaaabbde*
d	7	15840	*aaaaabbce*
e	11	10080	*aaaaabbcd*
aa	4	27720	*aaabbcde*
ab	6	18480	*aaaabcde*
ac	10	11088	*aaaabbde*
ad	14	7920	*aaaabbce*
ae	22	5040	*aaaabbcd*
bb	9	12320	*aaaaacde*
bc	15	7392	*aaaaabde*
bd	21	5280	*aaaaabce*
be	33	3360	*aaaaabcd*
cd	35	3168	*aaaaabbe*
ce	55	2016	*aaaaabbd*
de	77	1440	*aaaaabbc*
aaa	8	13860	*aabbcde*
aab	12	9240	*aaabcde*
aac	20	5544	*aaabbde*

aad	28	3960	*aaabbce*
aae	44	2520	*aaabbcd*
abb	18	6160	*aaaacde*
abc	30	3696	*aaaabde*
abd	42	2640	*aaaabce*
abe	66	1680	*aaaabcd*
acd	70	1584	*aaaabbe*
ace	110	1008	*aaaabbd*
ade	154	720	*aaaabbc*
bbc	45	2464	*aaaaade*
bbd	63	1760	*aaaaace*
bbe	99	1120	*aaaaacd*
bcd	105	1056	*aaaaabe*
bce	165	672	*aaaaabd*
bde	231	480	*aaaaabc*
cde	385	288	*aaaaabb*
aaaa	16	6930	*abbcde*
aaab	24	4620	*aabcde*
aaac	40	2772	*aabbde*
aaad	56	1980	*aabbce*
aaae	88	1260	*aabbcd*
aabb	36	3080	*aaacde*
aabc	60	1848	*aaabde*
aabd	84	1320	*aaabce*
aabe	132	840	*aaabcd*
aacd	140	792	*aaabbe*
aace	220	504	*aaabbd*
aade	308	360	*aaabbc*
abbc	90	1232	*aaaade*
abbd	126	880	*aaaace*
abbe	198	560	*aaaacd*
abcd	210	528	*aaaabe*
abce	330	336	*aaaabd*
abde	462	240	*aaaabc*
acde	770	144	*aaaabb*
bbcd	315	352	*aaaaae*
bbce	495	224	*aaaaad*
bbde	693	160	*aaaaac*
bcde	1155	96	*aaaaab*

aaaaa

aaaaa	32	3465	*bbcde*
aaaab	48	2310	*abcde*
aaaac	80	1386	*abbde*
aaaad	112	990	*abbce*
aaaae	176	630	*abbcd*
aaabb	72	1540	*aacde*
aaabc	120	924	*aabde*
aaabd	168	660	*aabce*
aaabe	264	420	*aabcd*
aaacd	280	396	*aabbe*
aaace	440	252	*aabbd*
aaade	616	180	*aabbc*

The ſame, ordered according to the greatneſs of the Numbers, will ſtand thus:

1	1	110880	*aaaaabbcde*
a	2	55440	*aaaabbcde*
b	3	36960	*aaaaabcde*
aa	4	27720	*aaabbcde*
c	5	22176	*aaaaabbde*
ab	6	18480	*aaaabcde*
d	7	15840	*aaaaabbce*
aaa	8	13860	*aabbcde*
bb	9	12320	*aaaaacde*
ac	10	11088	*aaaabbde*
e	11	10080	*aaaaabbcd*
aab	12	9240	*aaabcde*
ad	14	7920	*aaaabbce*
bc	15	7392	*aaaaabde*
aaaa	16	6930	*abbcde*
abb	18	6160	*aaaacde*
aac	20	5544	*aaabbde*
bd	21	5280	*aaaaabce*
ae	22	5040	*aaaabbcd*
aaab	24	4620	*aabcde*
aad	28	3960	*aaabbce*
abc	30	3696	*aaaabde*
aaaaa	32	3465	*bbcde*

be

be	33	3360	*aaaaabcd*
cd	. 35	. 3168	. *aaaaabbe*
aabb	36	3080	*aaacde*
aaac	40	2772	*aabbde*
abd	42	2640	*aaaabce*
aae	44	2520	*aaabbcd*
bbc	. 45	. 2464	. *aaaaade*
aaaab	48	2310	*abcde*
ce	55	2016	*aaaaabbd*
aaad	56	1980	*aabbce*
aabc	60	1848	*aaabde*
bbd	. 63	. 1760	. *aaaaace*
abe	66	1680	*aaaabcd*
acd	70	1584	*aaaabbe*
aaabb	72	1540	*aacde*
de	77	1440	*aaaaabbc*
aaaac	. 80	. 1386	. *abbde*
aabd	84	1320	*aaabce*
aaae	88	1260	*aabbcd*
abbc	90	1232	*aaaade*
aaaaab	96	1155	*bcde*
bbe	. 99	. 1120	. *aaaaacd*
bcd	105	1056	*aaaaabe*
ace	110	1008	*aaaabbd*
aaaad	112	990	*abbce*
aaabc	120	924	*aabde*
abbd	. 126	. 880	. *aaaace*
aabe	132	840	*aaabcd*
aacd	140	792	*aaabbe*
aaaabb	144	770	*acde*
ade	154	720	*aaaabbc*
aaaaac	. 160	. 693	. *bbde*
bce	. 165	672	*aaaaabd*
aaabd	168	660	*aabce*
aaaae	176	630	*abbcd*
aabbc	180	616	*aaade*
abbe	. 198	. 560	. *aaaacd*
abcd	210	528	*aaaabe*
aace	220	504	*aaabbd*

aaaaad

aaaaad	224	495	*bbce*
bde	231	480	*aaaaabc*
aaaabc .	240 .	462 .	*abde*
aabbd	252	440	*aaace*
aaabe	264	420	*aabcd*
aaacd	280	396	*aabbe*
aaaaabb	288	385	*cde*
aade .	308 .	360 .	*aaabbc*
bbcd	315	352	*aaaaae*
abce	330	336	*aaaabd*

20. Of Numbers (for inſtance) which have 12 Diviſors: To exhibit all the forms; and, all the Numbers in each form; not exceeding the Number 2048; (which is the loweſt Number of the higheſt form;) according to art. 15. 18.

All the ways according to which 12 may be expreſſed by Integers (as in art. 17, 18,) are $12 = 6 \times 2 = 4 \times 3 = 3 \times 2 \times 2$: Which affords us theſe forms, a^{11}, a^5b, a^3b^2, a^2bc. And in each of theſe, the Numbers are as follow; being in all 211.

a^{11}.

2048.

a^5b.

32	×	3	=	96
32	×	5	=	160
	×	7	=	224
	×	11	=	352
	×	13	=	416
	×	17	=	544
	×	19	=	608
	×	23	=	736
	×	29	=	928
	×	31	=	992
	×	37	=	1184
	×	41	=	1312
	×	43	=	1376
	×	47	=	1504
	×	53	=	1696
	×	59	=	1888
	×	61	=	1952
243	×	2	=	486
	×	5	=	1215
	×	7	=	1701

a^3b^2.

8	×	9	=	72
	×	25	=	200
	×	49	=	392
	×	121	=	968
	×	196	=	1352
27	×	4	=	108
	×	25	=	675
	×	49	=	1323
125	×	4	=	500
	×	9	=	1125

343	×	4	=	1372
a^2bc.				
4 × 3	×	5	=	60
	×	7	=	84
	×	11	=	132
	×	13	=	156
	×	17	=	204
	×	19	=	228
	×	23	=	276
	×	29	=	348
	×	31	=	372
	×	37	=	444
	×	41	=	492
	×	43	=	516
	×	47	=	564
	×	53	=	636
	×	59	=	708
	×	61	=	732
	×	67	=	804
	×	71	=	852
	×	73	=	876
	×	79	=	948
	×	83	=	996
	×	89	=	1068
	×	97	=	1164
	×	101	=	1212
	×	103	=	1236
	×	107	=	1284
	×	109	=	1308
	×	113	=	1356
	×	127	=	1524
	×	131	=	1572
	×	137	=	1644
	×	139	=	1668
	×	149	=	1788
	×	151	=	1812
	×	157	=	1884
	×	163	=	1956
	×	167	=	2004

4 × 5	×	7	=	140
	×	11	=	220
	×	13	=	260
	×	17	=	340
	×	19	=	380
	×	23	=	460
	×	29	=	580
	×	31	=	620
	×	37	=	740
	×	41	=	820
	×	43	=	860
	×	47	=	940
	×	53	=	1060
	×	59	=	1180
	×	61	=	1220
	×	67	=	1340
	×	71	=	1420
	×	73	=	1460
	×	79	=	1580
	×	83	=	1660
	×	89	=	1780
	×	97	=	1940
	×	101	=	2020
4 × 7	×	11	=	308
	×	13	=	364
	×	17	=	476
	×	19	=	532
	×	23	=	644
	×	29	=	812
	×	31	=	868
	×	37	=	1036
	×	41	=	1148
	×	43	=	1204
	×	47	=	1316
	×	53	=	1484
	×	59	=	1652
	×	61	=	1708
	×	67	=	1876
	×	71	=	1988

4×7	× 73	= 2044
4×11	× 13	= 572
	× 17	= 748
	× 19	= 836
	× 23	= 1012
	× 29	= 1276
	× 31	= 1364
	× 37	= 1628
	× 41	= 1804
	× 43	= 1892
4×13	× 17	= 884
	× 19	= 988
	× 23	= 1196
	× 29	= 1508
	× 31	= 1612
	× 37	= 1924
4×17	× 19	= 1292
	× 23	= 1564
	× 29	= 1972
4×19	× 23	= 1748
9×2	× 5	= 90
	× 7	= 126
	× 11	= 198
	× 13	= 234
	× 17	= 306
	× 19	= 342
	× 23	= 414
	× 29	= 522
	× 31	= 558
	× 37	= 666
	× 41	= 738
	× 43	= 774
	× 47	= 846
	× 53	= 954
	× 59	= 1062
	× 61	= 1098
	× 67	= 1206
	× 71	= 1278
	× 73	= 1314
9×2	× 79	= 1422
	× 83	= 1498
	× 89	= 1602
	× 97	= 1746
	× 101	= 1818
	× 103	= 1854
	× 107	= 1926
	× 109	= 1962
	× 113	= 2034
9×5	× 7	= 315
	× 11	= 495
	× 13	= 585
	× 17	= 765
	× 19	= 855
	× 23	= 1035
	× 29	= 1305
	× 31	= 1395
	× 37	= 1665
	× 41	= 1845
	× 43	= 1935
9×7	× 11	= 693
	× 13	= 819
	× 17	= 1071
	× 19	= 1197
	× 23	= 1449
	× 29	= 1827
	× 31	= 1953
9×11	× 13	= 1287
	× 17	= 1683
	× 19	= 1881
9×13	× 17	= 1989
25×2	× 3	= 150
	× 7	= 350
	× 11	= 550
	× 13	= 650
	× 17	= 850
	× 19	= 950
	× 23	= 1150
	× 29	= 1450

25×2 × 31 = 1550		49×2 × 17 = 1666	
× 37 = 1850		× 19 = 1862	
25×3 × 7 = 525		49×3 × 5 = 735	
× 11 = 825		× 11 = 1617	
× 13 = 975		× 13 = 1911	
× 17 = 1275		121×2 × 3 = 726	
× 19 = 1425		× 5 = 1210	
× 23 = 1725		× 7 = 1694	
25×7 × 11 = 1925		121×3 × 5 = 1815	
49×2 × 3 = 294		169×2 × 3 = 1014	
× 5 = 490		× 5 = 1690	
× 11 = 1078		289×2 × 3 = 1734	
× 13 = 1274			

These digested according to their natural order, ſtand thus:

60	306	516 ·	735	940
72	308	522	736	948 ·
84	315 ·	525	738	950
90	340	532	740 ·	954
96 ·	342	544	748 ·	968
108	348	550 ·	765	975
126	350	558	774	988 ·
132	352 ·	564	804	992
140	364	572	812 ·	996
150 ·	372	580	819	1012
156	380	585 ·	820	1014
160	392	608	825	1035 ·
198	414 ·	620	836	1036
200	416	636	846 ·	1060
204 ·	444	644	850	1062
220	460	650 ·	852	1068
224	476	666	855	1071 ·
228	486 ·	675	860	1078
234	490	693	868 ·	1098
260 ·	492	708	876	1125
276	495	726 ·	884	1148
294	500	732	928	1150 ·

1164

1164	1312	1524	1725	1924 ·
1180	1314	1550	1734 ·	1925
1184	1316	1564 ·	1746	1926
1196	1323 ·	1572	1748	1935
1197 ·	1340	1580	1780	1940
1204	1352	1602	1788	1952 ·
1206	1356	1612	1804 ·	1953
1210	1364	1617 ·	1812	1956
1212	1372 ·	1628	1815	1962
1215 ·	1376	1644	1818	1972
1220	1395	1652	1827	1988 ·
1236	1420	1660	1845 ·	1989
1274	1422	1665 ·	1850	2004
1275	1425 ·	1666	1854	2020
1276 ·	1449	1668	1862	2034
1278	1450	1683	1876	2044 ·
1284	1460	1690	1881 ·	2048
1287	1484	1694 ·	1884	
1292	1494 ·	1696	1888	
1305 ·	1504	1701	1892	
1308	1508	1708	1911	

Of the Numbers that are moſt convenient for the Purpoſe of dividing Large Quantities into Leſſer equal Parts.

21. Thoſe Numbers which (for the bigneſs of them) have the greateſt Number of Diviſors, and Aliquot Parts; have been wont to be made choice of, as moſt convenient for uſe; eſpecially when there may be frequent occaſion of dividing things ſo deſigned.

Hence it is, that the *Engliſh* Penny is divided into Four Farthings, (and almoſt all things in Four Quarters of a different Name,) becauſe there is often occaſion to divide into halves, and then again into halves. Hence alſo the *Roman* Pound,

1	1	1
2	2	a
4	3	a^2
6	4	ab
12	6	a^2b
24	8	a^3b
36	9	a^2bc
48	10	a^4b
60	12	a^2bc
120	16	a^3bc
180	18	a^2b^2c
240	20	a^4bc
360	24	a^3b^2c
720	30	a^4b^2c
840	32	a^3bcd
1260	36	a^2b^2cd
1680	40	a^4bcd
2520	48	a^3b^2cd
5040	60	a^4b^2cd
7560	64	a^3b^3cd
10080	72	a^5b^2cd
15120	80	a^4b^3cd
20160	84	a^6b^2cd
25200	90	$a^4b^2c^2d$
27720	96	a^3b^2cde
45360	100	a^4b^4cd
50400	108	$a^5b^2c^2d$
55440	120	a^4b^2cde
83160	128	a^3b^3cde
110880	144	a^5b^2cde
166320	160	a^4b^3cde
221760	168	$a^6b^2c^2de$
277200	180	$a^4b^2c^2de$
332640	192	a^5b^3cde
498960	200	a^4b^4cde
554400	216	$a^5b^2c^2de$
665280	224	a^6b^3cde

Pound, (and that which we now call the Pound *Troy Weight*,) is divided into 12 Ounces; and the *English* Shilling, into 12 pence; the Foot, into 12 Inches; the Zodiack, into 12 Signs; the Year, into 12 Months; becaufe, befide the Divifion into Quarters, it is further divifible by 3. And for a like reafon *Ptolemy* (and others after him) makes ufe of the Sexagenary divifion, of Integers into firft Minutes, or fmall, or *minute*, parts of the firft order; and of thefe, into Second Minutes, or Seconds, or fmall, or *minute*, parts of the Second order; and fo onward: becaufe 60 is divifible by 2 and 3, and likewife by 5. And the *Chinefes* (or *Cathaians*) Number their Years (and other things) by Revolutions of 60. After this; 360 is looked on as moft confiderable, becaufe it may be further divided by 2 and 3 once more: Which therefore is made the Number of Degrees in a Circle; admitting of 24 Divifors. And if this be not enough, each of thefe is divided into 60 Minutes; (that is, by 4, 3, 5, once more;) and thefe into Seconds, and fo forth. And the *Englifh* Pound Sterling, is divided into 20 Shillings; which Number is divifible by 4 and 5, (as 12, the Number of Pence in a Shilling, is divifible by 4 and 3;) which was accounted

counted more convenient than to make another Collection of Shillings by 12; becaufe this would not afford a divifion by 5. So that now 960 the Number of Farthings in a Pound Sterling, is for the firft ftep (from Farthings to Pence) divifible by 4; for the fecond ftep (from Pence to Shillings) by 4 and 3; for the third ftep (from Shillings to Pounds) by 4 and 5. And (without taking notice of the divifion of Pence into Farthings) the Number of Pence in a Pound Sterling, 240, is capable of 20 Divifors; and, of more than fo many, no Number is capable which is not greater than it.

Of the foregoing Table of Numbers in page 316.

In purfuance of which notion, I have here Collected a Table of all thofe Numbers, which (of all not greater than themfelves) have the greateft Number of Divifors; (together with the Number of Divifors in each of them, and the Form of their Compofition;) as far as 665,280, which hath 224 Divifors. All which (except 1,) are made by the Compofition of 2, 3, 5, 7, 11, (which I call *a*, *b*, *c*, *d*, *e*,) and the Powers of thefe, without admitting any other Prime. (But, if we would proceed to a greater Number of Divifors, we muft further take in $f = 13$.) And, of thefe, fome are of that nature, that none can have a greater Number of Divifors, which is not at leaft the double of them. Such are 1, 2, 6, 12, 60, 360, 2520: But not any after thefe for a great way.

Of

Of the Use of a Table of Prime Numbers.

22. For resolution of some of the Questions above mentioned, (as in art. 13, 14, 17, 18, &c,) it is very convenient to have at hand a Table of Prime Numbers: That we may know, by what Numbers to make trial of the Divisions therein directed. And, because, in great Numbers, it would be tedious to make trial of all the Prime Numbers in order, it is convenient also to know, by what Prime such greater Numbers may be divided.

In order to which, it is evident, in the first place, that all even Numbers may be divided by 2; and, if the Quotient of such division be even also, it may be again divided by 2, and so continually as long as the Quotient is an even Number.

It is also evident, in the second place, that all Numbers ending in 5, are divisible by 5; and, if they end in 0, then by 2 and 5. And so continually, as long as the Quotient of such division ends in 0, or 5.

It is known also, in the 3d place, that, if the Figures of any Number being added promiscuously (without regarding the places wherein they stand) are divisible by 9, (or, casting away 9 as oft as may be, nothing remains,) such Number is also divisible by 9. As in 29097; when (the Nines being left out, and) $2 + 7 = 9$ being cast away, nothing remains; whence we may conclude, 'tis divisible by $9 = 3 \times 3$. And I add further, as a fourth observation, (though I do not find that others have taken notice of it,) that the same holds also as to the Number 3: That is, from the Figures so promiscuously added, if 3 being cast away as oft as may be, nothing remain, such Number is divisible by 3; Otherwise, it is not. As in 530,967; where, all the threes, nines and sixes being left out (as manifestly divisible by 3,) the rest $5 + 7 = 12$, is so also, (or, (or, which is the ſame, 1 + 2 = 3;) ſo that all the threes being caſt away, nothing remains; whence we may conclude, that the whole Number is diviſible (though not by 9) at laſt by 3.

The ground of this and the former Obſervation is one and the ſame: Becauſe, the places increaſing in decuple proportion, if from 10, or any Number of tens, we caſt away all the nines or all the threes, there remains 1, or ſo many ones. So that, in caſe of ſuch caſting away of nines and threes, 1 and 10, have the ſame remainders; and ſo 2 and 20; 3 and 30, &c. And conſequently 1, 10, 100, 1000, &c, 2, 20, 200, 2000, &c. So that the ſame Figure, as to this, is of the ſame influence in whatever place it ſtand.

Of Dr. Pell's *Table of Prime Numbers.*

23. Beſide this, we have at the end of Dr. *Pell*'s *Algebra*, (Tranſlated and Publiſhed by *Thomas Branker*, in the Year 1668, with Dr. *Pell*'s directions,) a Compendious Table of all odd Numbers (not ending in 5) as far as 100,000; ſhewing not only, which of them are Prime Numbers; but alſo by what ſmalleſt Prime Number every other of them may be divided.

So that, whatever Number be propoſed, having divided it firſt by 2 and 5, (and if you will by 3 alſo,) as oft as may be, if it be capable of ſuch diviſion: If the reſult of ſuch diviſion do not exceed 100,000, we have direction in that Table, by what Prime it may be next divided; and then, by what Prime to divide the Quotient of ſuch Diviſion; and ſo continually, 'till we come to a Prime Number.

The reaſon why, in that Table, he omits all even Numbers, and all Numbers ending in 5, is obvious: to wit, Becauſe

Becaufe it appears to view (without the help of a Table) that fuch are accordingly divifible by 2, or 5.

He might, for a like reafon, have omitted alfo all that are divifible by 3, (becaufe this would prefently appear upon fuch promifcuous adding of the Figures as was but now mentioned ;) but that he could not well omit thefe, without difordering the Form of the Table.

Now, becaufe, in fuch Tables, it is of great moment that they be carefully Computed, and exactly Printed, (becaufe miftakes therein are not eafily obferved and Corrected by the Readers Eye,) I have taken care to examine that whole Table very exactly, (in the fame method and with the fame pains as if I were to compute it anew ;) and find that, though it had been Computed and Printed with great care, yet fome few miftakes (and but a few) have efcaped the Corrector's Eye. Moft of which are noted in the Table of *Errata*, Printed with it. Befide which I have obferved thefe that follow : Which (to fave another Reader the like labour) I have thought fit (for his eafe and fatisfaction) here to note. And, thefe being alfo amended as is here directed (befide thofe noted in the Printed *Errata*,) the Table will then be very accurate ; and (I think,) without any Error.

Pag.	*Numb.*	*For*	*Set*	*Pag.*	*Numb.*	*For*	*Set*
3	5579	P	7	28	55609	3	P
5	9287	19	37	31	60701	01	101
8	14873	73	107		60799	63	163
11	20983	3	P	33	64499	13	P
16	30167	71	97		65479	3	P
	31001	-29	29	34	67993	1	P
17	33409	47	P	38	75653	151	P
19	37583	13	7	41	80561	17	13
21	40049	19	29	43	85909	137	P
	40599	P	3	44	86993	79	P
	40759	3	P	47	93719	7	P
	41581	41	43	48	94769	41	97
24	46199	73	P	49	96109	3	13
27	53941	13	17		97487	3	13
28	54449	71	P				

Pag. 7, in the margin (after 43) for 37 fet 47.

By the help of this Table, if we had the Number propoſed 539,454,600, it is eaſy to reſolve it into the Primes of which it is Compoſed. For firſt, (becauſe of two Cyphers at the end) it is manifeſt that it may be divided twice by 2, and twice by 5. And then (becauſe theſe Cyphers being cut off, the Remainder is yet an even Number) it may be a third time divided by 2; and the reſult will be 2,697,273. And, if this Number were not beyond the reach of the Table, I ſhould ſeek it there; to ſee by what Prime it may be next divided. But, becauſe it is too big for it; I find, upon conſideration, that, the Figures being promiſcuouſly added, and 9 caſt away as oft as may be, nothing remains; and therefore that it may be divided by 9: Which being done, the next Quotient 299,697, may (for a like reaſon) be again divided (not by 9, but) by 3. And the Quotient 99899, is now come within the reach of this Table. And (without aſſaying the Prime Numbers 7, 11, 13, &c, in their order, till I come to a Prime Number by which it may be divided,) I find, by the Table, that it may be divided by 283, but not by any ſmaller Prime; and the Quotient of ſuch diviſion will be 353, another Prime. And therefore the Number propoſed 539,454,600 is $= 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 283 \times 353$.

But if, inſtead of 99,899, I had come to a Number greater than this Table, and yet not diviſible by 2, 5, or 3; I muſt then (for want of ſuch Table large enough) have been fain to make tryal of the conſequent Prime Numbers 7, 11, 13, &c, 'till by help of ſuch I had brought it within the Compaſs of the Table; And, if no ſuch can be found, before I come at a Prime as great as the Square Root of ſuch Number; I may then conclude ſuch Number to be a Prime.

CHAP. IV.

Monsieur FERMAT'S *Problems concerning Divisors and Aliquot Parts.*

IT is here proper to consider of such Questions (concerning Aliquot Parts) as those on account of which Monsieur *Fermat* and Monsieur *Frenicle* did value themselves; as is to be seen in my *Commercium Epistolicum*, Epist. 1, 11, 12, 22, 25, 26, 31, 33. And in a Treatise purposely Published on this occasion by Monsieur *Frenicle*, intituled, *Solutio duorum Problematum, circà numeros Cubos & Quadratos, quæ tanquam insolubilia universis Europæ Mathematicis à Clarissimo Viro D. Fermat sunt proposita*, &c, *à* D. B. F. D. B. *inventa*, &c, (that is, *à Domino Bernardo Frenicle de Bessy.*) *Parisiis apud Jacobum Langlois*, &c, 1657, in which he glories much that he was able to solve them. And amongst Monsieur *Fermat*'s *posthumous* Works, (Published since his death) the Publisher is pleased to insert his formal Challenge of me to solve them (with some others Letters to and from Monsieur *Fermat*, concerning the same) in these Words;

Problemata proposita à D. Fermat.

Proponatur (si placet) Wallisio, & reliquis Angliæ Mathematicis, sequens Quæstio Numerica.

Invenire Cubum, qui, additus omnibus suis partibus aliquotis, conficiat Quadratum. Exempli gratiâ, Numerus 343 *est Cubus à latere* 7. *Omnes ipsius partes aliquotæ sunt* 1, 7, 49, *quæ, adjunctæ ipsi* 343, *conficiunt numerum* 400, *qui est Quadratus à latere* 20. *Quæritur alius Cubus numerus ejusdem naturæ.*

Quæritur etiam numerus Quadratus, qui, additus suis partibus aliquotis, conficiat numerum Cubum.

Has

Has solutiones expectamus: Quas, si Anglia aut Galliæ Belgica & Celtica non dederint, Dabit Gallia Narbonensis; easque, in pignus nascentis amicitiæ, Domino Digby offeret & dicabit.

But was not so kind (though he there insert also divers Letters to and from Monsieur *Fermat*, concerning the same) as to insert those of mine, wherein I solved these (and others of) his Problems: Nor those of Monsieur *Fermat*, wherein he acknowledgeth that I had so done. Which are to be seen in my *Commercium Epistolicum*, in Epist. 23, 28, 29, 47, and elsewhere.

To those two Problems, I added a third of a like nature:

Invenire duos numeros Quadratos, qui, partibus suis aliquotis additi, eandem efficiant summam. Exempli gratiâ, $16 + 8 + 4 + 2 + 1 = 31 = 25 + 5 + 1$. *Inveniantur istiusmodi alii duo.*

The whole Mystery of solving these (and such like) Questions, I there discover in Epist. 23, which depends on what is here delivered in art. 8, 9, 10, 11, 12, of the Chapter here next preceeding.

For, 1. A Number added to all its Aliquot Parts, is all one as the Aggregate of its Divisors. 2. The Divisors of any Power of a Prime Number, (as of a) is a Geometrical Progression from 1 to such Power; as, for instance, of a^5, the Divisors are 1, a, aa, a^3, a^4, a^5. 3. And therefore the sum of such Geometrical Progression is the Aggregate of those Divisors. 4. This Aggregate is conveniently expressed by the Primes which Compose it. 5. The Divisors of any Power, or Degree, of one such Prime, severally Multiplied into all those of any Power, or Degree, of any other Prime, give all the Divisors of the Compound of those Powers. 6. And therefore the Aggregate of those first into the Aggregate of those second, give the Aggregate of the Divisors of such Compound. (For, by the common practice of Multiplication, all the Members of one Number, or Aggregate, Multiplied severally into all the Members of another, are equivalent to the whole of the one, multiplied into the whole of the other.) 7. And therefore the Primes Composing this laſt Aggregate, are the ſame with thoſe of both the Aggregates which Compoſe it. 8. And the ſame is in like manner to be argued, in caſe any Power, or Degree, of a third, fourth, or further, Prime, be continually Multiplied with thoſe foregoing: provided always, that they be all ſeveral Primes, and not any of the former Primes repeated; for, in ſuch caſe we are to follow the direction of art. 11, Chap. preceeding.

As, for inſtance; ſuppoſing $a = 2$, and therefore $a^5 = 32$: All the Diviſors hereof (or the Aggregate of ſuch Diviſors) are $1 + a + aa + a^3 + a^4 + a^5 = 1 + 2 + 4 + 8 + 16 + 32 = 63 = 3 \times 3 \times 7$. And ſuppoſing $b = 3$, and therefore $b^4 = 81$: The Aggregate of the Diviſors hereof are $1 + b + bb + b^3 + b^4 = 1 + 3 + 9 + 27 + 81 = 121 = 11 \times 11$: And therefore, of a^5b^4, the Aggregate of Diviſors is $63 \times 121 = 3 \times 3 \times 7, \times 11 \times 11$. And ſuppoſing further $c = 5$, and therefore $c^3 = 125$: The Aggregate of the Diviſors hereof are $1 + c + cc + c^3 = 1 + 5 + 25 + 125 = 156 = 2 \times 2 \times 3 \times 13$: And therefore, of $a^5b^4c^3$, the Aggregate of Diviſors is $63 \times 121 \times 156 = 3 \times 3 \times 7, \times 11 \times 11, \times 2 \times 2 \times 3 \times 13$, or $2 \times 2 \times 3 \times 3 \times 3 \times 7 \times 11 \times 11 \times 13$. And ſo onwards, in caſe of further Compoſitions.

Now, this being univerſal; it will be eaſy to make application thereof, to the particular caſes propoſed; or to any other of like nature.

As for Example.

I. The firſt Queſtion, is, *To find a Cube Number, which added to all its Aliquot Parts will make a Square*; (that is, the Aggregate of whoſe Diviſors ſhall be a Square Number.)

Here it is manifeſt, that ſuch Cube Number muſt be either the Cube of ſome Prime, (or at leaſt the ſecond, third, fourth, or further Cube of ſuch Prime; that is, ſome Power thereof whoſe exponent is diviſible by 3;) or elſe Compounded by the continual Multiplication of ſuch Cubes (firſt, ſecond, third, and ſo forth,) of two or more ſuch Prime Numbers. (For all ſuch, will be Cube Numbers, and no other but ſuch.)

Now

Now, if we can find any fuch Cube (firft, fecond, third, &c,) of any one Prime Number, whereof the Aggregate of Divifors being expreffed in Primes, thofe Primes will be all Pairs, (that is, each of them occurring an even Number of times;) fuch Aggregate ('tis manifeft) will be a Square Number; and therefore fuch Cube, will be fuch as is required.

And fuch Cube is $343 = 7 \times 7 \times 7$; whofe Divifors are $1 \times 7 \times 49 \times 343 = 400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5$; which is the Square of $2 \times 2 \times 5 = 20$.

When the Cubes (firft, fecond, third, or others,) of feveral Primes, have not their Aggregate of Divifors expreffable by Pairs of Primes; yet may the Compound of Two, Three, or more of fuch Cubes continually Multiplied (which will alfo be a Cube Number,) have its Aggregate of Divifors (which is the Compound of the feveral Aggregates continually Multiplied) fo expreffed: Namely, if the Cubes fo to be Compounded be fo chofen as that, what Primes in expreffing fome of the Aggregates be fingle, may be Paired by like fingle Primes in fome other of them.

Thus, for the Cube of 47, the Aggregate of Divifors (expreffed in Primes) is $2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 13 \times 17$; where (befide Pairs) we have 2, 3, 5, 13, 17; fingly: And, for the Cube of 5, the Aggregate is $2 \times 2 \times 3 \times 13$, where (befide Pairs) we have 3, 13, folitary; which (joined to thofe before) ferve to Pair 3, 13, but leave 2, 5, 17, yet folitary: And, for the Cube of 13, the Aggregate is $2 \times 2 \times 5 \times 7 \times 17$, which afford fellows to 5, 17, but leaves us 2, 7, yet folitary: And, for the Cube of 41, the Aggregate is $2 \times 2 \times 3 \times 7 \times 29 \times 29$; where (befide Pairs) we have 3, 7, folitary; which afford a fellow to 7, but leave 2, 3, folitary. So that for the Cube of $47 \times 5 \times 13 \times 41$, we have (befide Pairs) 2, 3, folitary. Which may thus be Paired.

For the Cube of 11, the Aggregate of Divifors is, $2 \times 2 \times 2 \times 3 \times 61$, where (befide Pairs) we have 2, 3, 61, folitary; which afford fellows to 2, 3, but leave 61, folitary: And, for the Cube of 27 (or the third Cube of 3, the Aggregate is $2 \times 2 \times 11 \times 11 \times 61$; which (befide Pairs) affords a fellow to 61. So that, for the Cube of $47 \times 5 \times 13$

$\times 41$

×41×11×27, (or 27×5×11×13×41×47) the Aggregate of Divisors, is 2×2×2×2×2×3×5×13×17, ×2×2×3×13, ×2×2×5×7×17, ×2×2×3×7×29×29, ×2×2×2×3×61, ×2×2×11×11×61: Or (putting the Primes in order) 2×2×2×2×2×2×2×2×2×2×2×2×2×2×2×2×3×3×3×3×5×5×7×7×11×11×13×13×17×17×29×29×61×61; where we have 2, sixteen times; 3, four times; and 5, 7, 11, 13, 17, 29, 61, twice; which therefore (being all continually Multiplied) must needs afford a Square Number. Which was the thing required to be found in Monsieur Fermat's first Question.

In like manner; if with the Cube of 47×5×13×41 (as before) we Compound the Cubes of 2, and of 3, where we have the Aggregates 3×5, and 2×2×2×5, which (beside Pairs) afford us 2, 3, solitary; which afford fellows to 2, 3, that were solitary before. And therefore for the Compound Cube of 47×5×13×41×2×3 (or 2×3×5×13×41×47) we shall have (in the Compound Aggregate of Divisors) these Primes Components, 2, fourteen times; 3 and 5, four times; 7, 13, 17, and 29, twice: Which being all continually Multiplied will also make a Square Number. Which was the thing required to be found in Monsieur Fermat's first Question.

These two Compound Cubes, if they be further Compounded with the Cube of 7 (which is no ingredient in either of them) will afford two more; whose Aggregate of Divisors will (beside the Primes in each of them respectively) have these farther Primes Components, 2, four times; and 5, twice: Which, being Compounded with the fore-mentioned Squares, will still afford Square Numbers.

So have we five Cubes, whose Aggregate of Divisors are Squares.

Roots of the Cubes.

7.
27 × 5 × 11 × 13 × 41 × 47.
2 × 3 × 5 × 13 × 41 × 47.
27 × 5 × 7 × 11 × 13 × 41 × 47.
2 × 3 × 5 × 7 × 13 × 41 × 47.

Roots

Roots of the Squares.

2×2×5.
2× (Eight-times) ×3×3×5×7×11×13×17×29×61.
2× (Seven-times) ×3×3×5×5× 7×13×17×29.
2× (Ten-times) ×3×3×5×5× 7×11×13×17×29×61.
2× (Nine-times) ×3×3×5×5× 5× 7×13×17×29.

In all which I make uſe of no Cube of a Prime which is not leſs than 100. And, in like manner, may other ſuch Cubes be found; as is there ſhewed in Epiſt. 23, and 28. Such as theſe :

Roots of the Cubes.

2×3×5×13×17×31×41×191.
2×3×5× 7×13×17×31×41×191.
3×3×3× 5×11×13×17×31×41×191.
3×3×3× 5× 7×11×13×17×31×41×191.
17×31×47×191.
7×17×31×47×191.

Roots of the Squares.

2×(Twelve-times) 3×3×3×5×5×7×13×17×29×29×37.
2×(Fourteen-times)3×3×3×5×5×5×7×13×17×29×29×37.
2×(Thirteen-times)3×3×3×5×7×11×13×17×29×29×37×61.
2×(Fifteen-times) 3×3×3×5×5×7×11×13×17×29×29×37×61.
2×(Ten-times) 3×3×5×13×17×29×37.
2×(Twelve-times) 3×3×5×5×13×17×29×37.

In all which I make uſe of no Cube of a Prime Number which is not leſs than 200.

But, in order to make theſe Inquiries for ſuch Cubes; it is expedient to have at hand a Table of the Cubes of Prime Numbers (and of the ſecond, third, or further Cubes, of the leſſer of them,) or of the Roots of ſuch Cubes; with the

the Aggregate of Divisors (in each of those Cubes) expressed in Primes.

And, to save the Reader the labour of computing such a-new, I here subjoin what I have at hand.

Roots of the Cubes.	Aggregate of their Divisors.
1	1
2	3 × 5
4	127
8	3 × 11 × 31
16	8191
32	3 × 5 × 17 × 257
3	2 × 2 × 2 × 5
9	1093
27	2 × 2 × 11 × 11 × 61
81	797161
243	2 × 2 × 2 × 2 × 2 × 5 × 17 × 41 × 193
5	2 × 2 × 3 × 13
25	19531
125	2 × 3 × 11 × 71 × 521
7	2 × 2 × 2 × 2 × 5 × 5
11	2 × 2 × 2 × 3 × 61
13	2 × 2 × 5 × 7 × 17
17	2 × 2 × 3 × 3 × 5 × 29
19	2 × 2 × 2 × 5 × 181
23	2 × 2 × 2 × 2 × 3 × 5 × 53
29	2 × 2 × 3 × 5 × 421
31	2 × 2 × 2 × 2 × 2 × 2 × 13 × 37
37	2 × 2 × 5 × 2603
41	2 × 2 × 3 × 7 × 29 × 29
43	2 × 2 × 2 × 5 × 5 × 11 × 37
47	2 × 2 × 2 × 2 × 2 × 3 × 5 × 13 × 17
53	2 × 2 × 3 × 3 × 3 × 5 × 281
59	2 × 2 × 2 × 3 × 5 × 1741
61	2 × 2 × 31 × 1861
67	2 × 2 × 2 × 5 × 17 × 449
71	2 × 2 × 2 × 2 × 3 × 3 × 2521
73	2 × 2 × 5 × 13 × 37 × 41

Roots

Roots of the Cubes.	Aggregates of their Divisors.
79	2 × 2 × 2 × 2 × 2 × 5 × 3121
83	2 × 2 × 2 × 3 × 5 × 7 × 13 × 53
89	2 × 2 × 3 × 3 × 5 × 17 × 233
97	2 × 2 × 5 × 7 × 7 × 941
101	2 × 2 × 3 × 17 × 5101
103	2 × 2 × 2 × 2 × 5 × 13 × 1061
107	2 × 2 × 2 × 3 × 3 × 3 × 5 × 5 × 229
109	2 × 2 × 5 × 11 × 13 × 457
113	2 × 2 × 3 × 5 × 19 × 1277
127	2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 5 × 1613
131	2 × 2 × 13 × 19 × 2293
137	2 × 2 × 3 × 5 × 23 × 1877
139	2 × 2 × 2 × 5 × 67627
149	2 × 2 × 3 × 5 × 5 × 11 × 101
151	2 × 2 × 2 × 2 × 13 × 19 × 877
157	2 × 2 × 5 × 5 × 17 × 29 × 79
163	2 × 2 × 41 × 2657
167	2 × 2 × 2 × 3 × 7 × 2789
173	2 × 2 × 3 × 5 × 29 × 41 × 73
179	2 × 2 × 2 × 3 × 3 × 5 × 37 × 433
181	2 × 2 × 7 × 13 × 16381
191	2 × 2 × 2 × 2 × 2 × 2 × 2 × 3 × 17 × 29 × 37
193	2 × 2 × 5 × 5 × 5 × 14453
197	2 × 2 × 3 × 3 × 5 × 42691
199	2 × 2 × 2 × 2 × 5 × 5 × 19801

If, in the Queſtion propoſed, it had been required that the Aggregate of Diviſors (of the Cube ſought) ſhould be (not a Square Number, but) the *Double*, *Treble*, or otherwiſe *Multiple*, of a Square Number: The proceſs would be juſt the ſame, (and the ſame Table will ſerve,) ſave that, then, the Aggregate is to be diviſible by 2, 3, or ſuch other Number as is the exponent of the propoſed Multiple, and the reſt of the Primes compoſing it to be all Pairs.

Thus, if the *Decuple* of a Square be required; the Cube of 3 will anſwer it; where the Aggregate is 2 × 2 × 2 × 5; that is, beſide 2 × 5 = 10, the other Components are Pairs.

If the *Quadruple* of a Square (which muſt therefore itſelf be be a Square;) the Cube 7 Anſwers it; whoſe Aggregate is $2 \times 2 \times 2 \times 2 \times 5 \times 5$: Out of which, if we exempt $2 \times 2 = 4$, the reſt are Pairs. And ſo will any other Cube whoſe Aggregate of Diviſors is an even Square, and therefore diviſible by 4.

If the *Sextuple* be required: The Cube of 27×11 anſwers it; where the Aggregate is $2 \times 2 \times 11 \times 11 \times 61$, $\times 2 \times 2 \times 2 \times 3 \times 61$. Whence if we exempt $2 \times 3 = 6$, the reſt are Pairs: And ſo will alſo (for the ſame reaſon) the Cube of 2×3; where the Aggregate is 3×5, $\times 2 \times 2 \times 2 \times 5$. And the like in other caſes.

But if ſuch Multiple ſhould be required, as that no Aggregate can be found (or not within certain limits) which, being divided by the Exponent of that Multiple, will leave the reſt of the Prime Components Pairs; ſuch caſe (at leaſt within ſuch limits) is an impoſſible caſe.

As, if we demand a Square's Multiple by 23, 43, or 47; and confine ourſelves to the Cubes of the Table foregoing; it is manifeſt that (without aſſuming the Cube of ſome other Prime, or ſome further Cube of ſome of theſe,) it cannot be done. For here, amongſt all the Prime Components of the Aggregates, the Numbers 43, and 47, come not at all; and though 23 come once (at the Cube of 137) yet it is there joyned with 1877, which (coming no more) cannot be Paired by any ſuch Compoſition of the propoſed Aggregates. (Remembring always, what was before noted, that the Aggregates for two or more Cubick Powers of the ſame Prime, are not here to be Compounded.) So that (within the limits of the Table) the caſe is not poſſible. And the like may be ſhewed of many others: I ſay, not poſſible *within the limits of this Table.* But, to ſay it is *not at all poſſible*, through the whole extent of all poſſible Numbers; is (I think) too bold an aſſertion for any to make out.

Of the Second Queſtion propoſed by Monſieur FERMAT. *See above, page* 322.

II. The Second Queſtion is, (*To find a Square Number, which*

which added to all its Aliquot Parts will make a Cube; that is, the Aggregate of whoſe Diviſors ſhall be a Cubick Number.)

And here the proceſs is much the ſame as before; ſave that here we ſhall need a Table of Square Numbers, (as there of Cubes,) with their Aggregate of Diviſors expreſſed in Primes: And here we are to find out, or ſo to Compound, the Aggregates, as that the Primes expreſſing them may be (not Couples or Duplicates, as there, but) Triplicates: That is, that each Prime may occur three, ſix, nine, or other Number of times diviſible by three.

But, though the proceſs be much the ſame, yet the ſucceſs will not be altogether ſo ready as there; becauſe Triplicates of the Components will not be ſo eaſily adjuſted as Duplicates. (And, for the ſame reaſons, if Biquadrates, or Surſolids, or ſome higher Powers, were required; the proceſs would ſtill be much the ſame, but the trouble of finding ſuch would ſtill be increaſed.)

Such Table of Squares (becauſe I have it at hand) I ſhall here ſubjoin; to ſave the Reader (who ſhall think fit to give himſelf the trouble of inquiring into ſuch Queſtions) the labour of Computing the ſame again.

Roots of the Squares.	Aggregate of their Diviſors.
1	1
2	7
4	31
8	127
16	511 = 7 × 73
32	2047 = 23 × 89
64	8191
128	32767 = 7 × 31 × 151
256	131071
3	13
9	121 = 11 × 11
27	1093
81	9841 = 13 × 757
243	88573 = 23 × 3851

 Roots

Roots of the Squares.	Aggregate of their Divisors.
5	31
25	781 = 11 × 71
125	19531
625	488281 = 19 × 31 × 829
7	57 = 3 × 19
49	2801
343	137257 = 29 × 4733
2401	6725601 = 3 × 3 × 19 × 37 × 1063
11	133 = 7 × 19
121	16105 = 5 × 3221
13	183 = 3 × 61
169	30941
17	307
289	88741
19	381 = 3 × 127
361	137561 = 151 × 911
23	553 = 7 × 79
29	871 = 13 × 67
31	993 = 3 × 331
37	1407 = 3 × 7 × 67
41	1723
43	1893 = 3 × 631
47	2257 = 37 × 61
53	2863 = 7 × 409
59	3541
61	3783 = 3 × 13 × 97
67	4557 = 3 × 7 × 7 × 31
71	5113
73	5403 = 3 × 1801
79	6321 = 3 × 7 × 7 × 43
83	6973 = 19 × 367
89	8011
97	9507 = 3 × 3169
101	10303
103	10713 = 3 × 3571
107	11557 = 7 × 13 × 127
109	11991 = 3 × 7 × 571

Roots

Roots of the Squares	Aggregate of their Divisors.
113	12883 = 13 × 991
127	16257 = 3 × 5419
131	17293
137	18907 = 7 × 37 × 73
139	19461 = 3 × 13 × 499
149	22351 = 7 × 3193
151	22953 = 3 × 7 × 1093
157	24807 = 3 × 8269
163	26733 = 3 × 7 × 19 × 67
167	28057
173	30103
179	32221 = 7 × 4603
181	32943 = 3 × 79 × 139
191	36673 = 7 × 13 × 13 × 31
193	37443 = 3 × 7 × 1783
197	39007 = 19 × 2053
199	39801 = 3 × 13267
211	44733 = 3 × 13 × 31 × 37
223	49953 = 3 × 16651
227	51757 = 73 × 709
229	52671 = 3 × 97 × 181
233	54523 = 7 × 7789
239	57361 = 19 × 3019
241	58323 = 3 × 19441
251	63253 = 43 × 1471
257	66307 = 61 × 1087
263	69433 = 7 × 7 × 13 × 109
269	72631 = 13 × 37 × 151
271	73713 = 3 × 24571
277	77007 = 3 × 7 × 3667
281	79243 = 109 × 727
283	80373 = 3 × 73 × 367
293	86143
307	94557 = 3 × 43 × 733
311	97033 = 19 × 5107
313	98283 = 3 × 181 × 181
317	100807 = 7 × 14401

Roots

Roots of the Squares.	Aggregate of their Divifors.
331	109893 = 3 × 7 × 5233
337	113907 = 3 × 43 × 883
347	120757 = 7 × 13 × 1327
349	122151 = 3 × 19 × 2143
353	124963 = 19 × 6577
359	129241 = 7 × 37 × 499
367	135037 = 7 × 101 × 191
373	139503 = 3 × 7 × 7 × 13 × 73
379	144021 = 3 × 61 × 787
383	147073
389	151711 = 7 × 21673
397	158007 = 3 × 31 × 1699
401	161203 = 7 × 23029
409	167691 = 3 × 55897
419	175981 = 13 × 13537
421	177663 = 3 × 59221
431	186193 = 7 × 67 × 397
433	187923 = 3 × 37 × 1693
439	193161 = 3 × 31 × 31 × 67
443	196693
449	202051 = 97 × 2083
457	209307 = 3 × 7 × 9967
461	213083 = 13 × 37 × 443
463	214833 = 3 × 19 × 3769
467	218557 = 19 × 11503
479	229921 = 43 × 5347
487	237657 = 3 × 7 × 11317
491	241573 = 37 × 6529
499	249501 = 3 × 7 × 109 × 109

Now it is manifeft, upon view, that (if we confine ourfelves to the limits of this Table) many of thefe Numbers are not of ufe to the prefent purpofe. Becaufe many of the Primes (amongft the Aggregates) come but once; as 5. 29. 71. 89. 101. 139. 191. 307. 331. 397. 409. 443. 571. 631. 709. 727. 733. 757. 787. 829. 883. 911. 991. 1063. 1087. 1327. 1471. 1693. 1699. 1723. 1783. 1801. 2053. 2083. 2143.

2143. 2801. 3019. 3169. 3193. 3221. 3541. 3571. 3667. 3769. 3851. 4603. 4733. 5107. 5113. 5233. 5347. 5419. 6529. 6577. 7789. 8011. 8191. 8269. 9967. 10303. 11317. 11503. 13267. 13537. 14401. 16651. 17293. 19441. 19531. 21673. 23029. 24571. 28057. 30103. 30941. 55897. 59221. 86143. 88741. 131071. 147073. 196693. Others but twice (not thrice) as 23. 79. 367. 499. 1093. And therefore cannot by any Compoſition (within theſe limits) make a Cube. And, conſequently, all the Squares to which any of them belong, are to be laid aſide as not of uſe. And thoſe are, the Squares of 32, 64, 256, 27, 81, 243, 25, 125. 625. 49, 343, 2401, 121, 169, 17, 289, 361, 23, 31, 41, 43, 53, 59, 71, 73, 83, 89, 97, 101, 103, 109, 113, 127, 131, 139, 149, 151, 157, 167, 173, 179, 181, 193, 197, 199, 223,. 227, 233, 239, 241, 251, 257, 271, 277, 281, 283, 293, 307, 311, 317, 331, 337, 347, 349, 353, 359, 367, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 443, 449, 457, 461, 463, 467, 479, 487, 491. (And the Square of 1, is, in this caſe, inſignificant; becauſe a Multiplication by 1 makes no alteration.) And, theſe being laid aſide, we muſt alſo lay aſide the Squares of 128, 9, 13, 47, 61, 79, 229, 269. Becauſe, in thoſe that remain, 43 occurs but once; and 11, 61, 97, 151, but twice. And, thoſe being laid aſide, we muſt alſo lay aſide the Squares of 137, 211, 313, becauſe, in thoſe now remaining, 37, 181, occur but twice. And (137 being laid aſide) the Squares of 16, 373, muſt alſo be laid aſide; becauſe now 73 comes but twice.

So that we have now but theſe few left for conſideration, to wit, the Squares of 2, 4, 8, 3, 5, 7, 11, 19, 29, 37, 67, 107, 163, 191, 263, 439, 499. Which, with their Aggregates, ſtand thus:

2	7	5	31	29	13 × 67	163	3 × 7 × 19 × 67
4	31	7	3 × 19	37	3 × 7 × 67	191	7 × 13 × 13 × 31
8	127	11	7 × 19	67	3 × 7 × 7 × 31	263	7 × 7 × 13 × 109
3	13	19	3 × 127	107	7 × 13 × 127	439	3 × 31 × 31 × 67
						499	3 × 7 × 109 × 109

In

In which there is no Prime (amongſt the Aggregates) which doth not occur at leaſt three times. That is, 3 ſeven times; 7 eleven times; 13 and 31 ſix times; 67 four times; 19, 109, 127, three times.

Of theſe I will firſt conſider 127; which, becauſe it comes but thrice, we muſt take all or none of them. If all, then this (at 107) brings in 13; which muſt therefore be trebled. And it muſt be done one of theſe three ways, either by taking in the Squares of 3 and 29; or of 3 and 263; or of 191 alone.

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31
499	3, 7, 109, 109
263	7, 7, 13, 109
191	7, 13, 13, 31

If the firſt way, this (at 29) brings in 67. Which (that it may be trebled) brings in two of theſe 3 Squares 37, 163, 439. Of which, if 163 be one, this (becauſe of 19) brings in the Squares 7 and 11. And if, for the other, we take the Square of 37; this brings in 3 and 7 a fourth time, and therefore either each of them muſt come in twice more (that we may have them ſix times) or elſe 37 muſt here be laid aſide. Now if, for 3 twice, we take (for one of them) the Square of 439, this brings in a fourth 67; which muſt not be (unleſs we could have it ſix times, which we cannot.) Therefore, if at all, this 3 twice, muſt be ſupplied by the Squares of 67 and 499 (for there is no other ſupply;) which brings in 109 twice; and this (that it may be tripled) requires the Square of 263. But, with this, comes in 13 a fourth time; and therefore (that we may have it ſix times) we muſt take in the Square of 191. But, by this time, we have 7 ten times; which muſt not be, unleſs we could (which we cannot) have it twelve times. Therefore the Square of 37 muſt here be laid aſide. If then (retaining that of 163) we take (inſtead of 37) the Square of 439; this brings in 3 a fourth time; which therefore we muſt have twice more. But not from the Square of 37 (becauſe already laid by, and becauſe it would bring in a fourth 67;) therefore,

therefore, if at all, from the Squares of 67 and 499 (as before,) which requires that of 263; and, this, that of 191, as before. But now we have 31 a fourth time, which requires it twice more; which is not to be had, ſave at the Squares of 4 and 5; whereof that of 4 is not to be admitted, as being included in that of 8 already taken. So that the Square of 163 cannot be taken either with that of 37 or of 439, and muſt therefore be laid aſide; (and, with it, the Squares of 7 and 11.) And conſequently (retaining that of 3 and of 29,) we muſt (for trebling of 67) take the Squares of 37 and 439. And here we have 31 twice, and muſt therefore have it a third time: But not from the Square of 4; (becauſe included in that of 8:) Therefore either from that of 5, or of 191. If from that of 5; we ſhall want a third 7 (having yet but two;) which we cannot have from the Square of 2 (becauſe included in 8;) nor from 163 (becauſe already rejected;) nor from that of 11 (becauſe already excluded with that of 163;) nor from that of 191, becauſe this would bring in a fourth 31, (which may not be, becauſe we cannot have it ſix times without the Square of 4, which is included in that of 8;) nor from that of 69 (for the ſame reaſon;) nor from that of 499, becauſe this cannot ſtand without that of 263; nor from both theſe together; becauſe then we ſhall have it five times, but cannot have it a ſixth; (all the reſt wherein 7 is found, being already excluded.) Therefore (omitting that of 5) we muſt (if at all) have a third 31 from the Square of 191. But this brings in a fourth and fifth 13; which (for a ſixth) will require the Square of 263; and this (becauſe of 109) the

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
439	3, 31, 31, 67
67	3, 7, 7, 31
499	3, 7, 109, 109
263	7, 7, 13, 109
191	7, 13, 13, 31
4	31
5	31

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
5	31

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
191	7, 13, 13, 31
263	7, 7, 13, 109
499	3, 7, 109, 109

Square of 499. And this (beside Triplicates) brings in a fourth 3; (which therefore will afford, not a Cube, but the Triple of a Cube, if that had been required;) we want therefore 3 twice more (to make it up six times;) but can have neither of them from the Squares of 7 or 163 (as being already excluded,) nor from that of 67, (as bringing in a fourth 31,) and therefore not at all. And, consequently, this first way (by the Squares of 3 and 29) doth not succeed.

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
7	3, 19
163	3, 7, 19, 67
11	7, 19
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31

The second way of supplying 13 twice, (which at the Square of 107 were wanting;) is, from the Squares of 3 and 263: Which (because of 109) requires that of 499. And, because (amongst the Aggregates) we have 3 twice; we must have it a third time. If, for this, we take in the Square of 7, or of 163; either of these (because of 19) brings in the other, and that of 11. And now, because of 67 once, we must have it twice more. But not from the Square of 29 (being already excluded as not to be taken with that of 3;) and therefore from the Squares of 37, and 439. And, by this time we have 3 six times (and more than so, we may not have it, unless we could have it nine times;) and 7 we have 7 times, and therefore must have it twice more: But, not from the Square of 2 (as being included in that of 8;) nor from that of 191, (because this would bring in 13 a fourth and a fifth time, which would require a sixth, from the Square of 29 already rejected;) therefore, if at all, from the Square of 67. But neither can this be, (because it brings in a seventh 3; which may not be, there being no more to make it up nine times:) And, consequently, the third 3 (wanting at the Square of 499) is not to be supplied

ſupplied from the Squares of 7, or of 163. If then (omitting theſe two) we ſhould take (for a third 3) the Square of 37 or of 439, either of theſe (becauſe of 67) would bring in the other, and alſo require that of 29, or of 163, already rejected. If then (omitting theſe of 37 and 439) we take (for a third 3) the Square of 67; this brings in 31, which is therefore to be Tripled. But not from the Square of 4 (as included in that of 8;) nor from the Square of 191 (becauſe that would bring in a fourth and fifth 13, which would require a ſixth from the Square of 29 already rejected;) nor from the Square of 439 (becauſe of 67 there, which would bring in that of 29, or 37, or 163, already rejected) nor from the Square of 5, becauſe (though that would afford a ſecond 31,) a third would yet be wanting, and not to be had. And, conſequently, (there being no other place from whence to fetch a third 3) this ſecond way will not ſucceed.

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
37	3, 7, 67
439	3, 31, 31, 67

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
67	3, 7, 7, 31

The third way for ſupplying 13 twice, (which at the Square of 107 were wanting) is (omitting the Squares of 3, 29, 263,) from the Square of 191. And, becauſe here we have 31 once, this muſt be Tripled. But not from the Square of 4: (as included in 8;) And therefore, if at all, either from that of 439 (where it is twice,) or from the Squares of 5 and 67. If from that of 439; then 67 (here found) muſt be Tripled; but not from the Square of 29 (as already excluded,) therefore from thoſe of 37, and 163; and this laſt (becauſe of 19) calls in thoſe of 7 and 11. But, by this time, we have 3 five times, and therefore

8	127
19	3, 127
107	7, 13, 127
191	7, 13, 13, 31
439	3, 31, 31, 67
37	3, 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31

ſhould have it a ſixth time; but not from the Square of 499 (for that would recall that of 263 already rejected;) therefore, if at all, from that of 67; but we ſhall then have 7 ſeven times; which is not to be admitted, ſince we cannot have it nine times. Therefore (omitting that of 439, and therefore thoſe of 37 and 163) take we thoſe of 5 and 67.

8	127
19	3, 127
107	7, 13, 127
191	7, 13, 13, 31
5	31
67	3, 7, 7, 31

And, by this time; we have 7 four times; and therefore, if at all, we muſt have it twice more. But not from the Square of 2 (as included in 8;) nor from that of 37 or 163 (as already rejected, with that of 439;) nor from that of 11 (which, becauſe of 19, would bring us back to that of 163 already rejected;) nor from 499 (which, becauſe of 109, would bring us back to that of 263 already laid aſide;) and therefore not at all. So that this third way fails alſo: And, conſequently, the Squares of 8, 19, 107, (where we meet with 127,) muſt all be laid aſide.

We have then but theſe left to be further conſidered.

2	7	5	31	29	13, 67	163	3,7,19,67	439	3,31,31,67
4	31	7	3,19	37	3,7,67	391	7,13,13,31	499	3,7,109,109
3	13	11	7,19	67	3,7,7,31	263	7,7,13,109		

And here we will begin with the Prime 109; which, becauſe it comes but once at the Square of 263, and twice at that of 499; theſe muſt either both be taken, or both omitted.

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31
439	3, 31, 31, 67

And becauſe, in theſe, we have 13 once; this muſt be taken twice more. And therefore either from the Squares of 3 and 29, or from that of 191 above; (ſince we have it now but five times in all.)

If the firſt way; then, becauſe of 67 once, we muſt take it twice more; from two Squares of theſe three, 37, 163, 439. Firſt, let thoſe be the Squares

Squares of 37 and 163; therefore (becauſe of 19) we muſt take alſo thoſe of 7 and 11. And, by this time, we have 3 four times, (and this affords us, not a Cube, but the Triple of a Cube, if that were required;) we muſt therefore take it twice more; which is only to be had at the Squares of 67 and 439, (for now we have it but ſix times in all,) but this brings in a fourth 67 which cannot be admitted. Secondly, let it be the Squares of 37 and of 439: which brings in 31 twice, and we muſt therefore have it a third time. Which if we take from the Square of 67; this brings in a fourth 3; which will require two more, from the Squares of 7 and 163; which will bring in a fourth 67. If from the Square of 191; this brings in a fourth and fifth 13, which cannot be admitted, becauſe we have not a ſixth. If from the Square either of 4, or of 5; either of theſe (beſide Triplicates) would leave us 7 four times (which would afford, not a Cube, but the Septuple of a Cube, if that had been required;) but this requires 7 twice more. Neither of which can be had from the Squares of 67, or 191, (as being already rejected;) nor from that of 163 (as bringing in a fourth 67;) and therefore, if at all, from the Squares of 2 and 11. But this would bring in 19; and therefore (to Triple it) will call in the Squares of 7 and 163; (which laſt is already rejected, and would bring in a fourth 67;) therefore not at all. Thirdly, (omitting that of 37) let this 67 twice, be taken from the Squares of 163 and 439. But this (becauſe of 19) calls in the Squares of 7 and 11; and conſequently, (becauſe then we have 3 four times) the Squares of 37 and 67 already rejected. So that this firſt way ſucceeds not.

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31
7	3, 19
163	3, 7, 19, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 3, 67
191	7, 13, 13, 31

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
4, 5	31
2	7
11	7, 19
7	3, 19
163	3, 7, 19, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
163	3, 7, 19, 67
439	3, 31, 31, 67
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
439	3, 31, 31, 67
37	3, 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31
4	31
5	31

If we take the ſecond way of ſupplying 13 twice, (which at the Squares of 263 and 499 were wanting) by the Square of 191 (omitting thoſe of 3 and 29;) then, becauſe here we have 31 once, which muſt therefore be ſupplied twice more: We will firſt try whether it may be done by the Square of 439 (where it comes twice;) and then whether it can be done without this.

If we ſupply it from the Square of 439; this brings in 67, which muſt therefore be Tripled: But not by the Square of 29 (as already rejected, and as bringing in a fourth 13;) therefore from thoſe of 37 and 163. Where becauſe we have 19 once, we muſt have it twice more, from the Squares of 7 and 11. And by this time we have 7 ſeven times, and muſt therefore have it twice more: And we have 3 five times, and muſt therefore have it once more. Both which we may have from the Square of 67 (and from thence only, becauſe 3 is to be had no where elſe;) and now we have 31 a fourth time; which requires it twice more (that it may be ſix times;) and theſe we have at the Squares of 4 and 5. So that now we have a Cube compleated; whoſe Components are, 7, nine times; 3 and 31, ſix times; 13, 67, and 109, three times. And the Square whence it ariſeth, is that of 4 × 5 × 7 × 11 × 37 × 67 × 163 × 191 × 263 × 439 × 499.

2	7
3	13
29	13, 67

The

The remaining Squares which are not ingredients into this, are thoſe of 2, 3, 29.

Now if from theſe (without the other) we could form another Cube, ſuch Cube would not only be another ſuch Cube as is deſired, but (being a Prime to that already found) might be Compounded with that found, to make a third. But this cannot be: Becauſe (for theſe) we have no Prime that comes three times.

It remains to ſee, if (omitting the Square of 439) we can otherwiſe ſupply 31 twice, which at the Square of 191 were wanting. Where, firſt, it is manifeſt, that (the Square of 439 being laid aſide) thoſe of 37 and 163 (becauſe of 67) muſt alſo be laid aſide, unleſs we can have a third 67 from the Square of 29. Which cannot be, becauſe this would introduce a fourth 13, and we have not two more to make up ſix. Then, having laid by that of 163, we muſt (becauſe of 19) lay by thoſe of 7 and 11. So that there remain only the Squares of 2, 4, 5, 67, to ſupply 31 twice (becauſe we have it once) and 7 twice (becauſe we have it four times) and 3 twice (becauſe we have it once.) Now 31 might be ſupplied twice from the Squares of 4 and 5, (but then we could take no more, becauſe that of 2 is included in 4; and 67 would bring in a fourth 31.) Or it might be ſupplied by one of thoſe (ſuppoſe 5,) with that of 67. And thus we ſhould have a ſupply of 31 twice, and of 7 twice, and of 3 once: But there wants another 3 (which the remaining Squares of 2 and 4 cannot ſupply) to compleat the Cube. So that this affords, not a Cube, but $\frac{1}{3}$ of a Cube. There is therefore no other Cube (but that before aſſigned) here to be had, retaining (as is hitherto ſuppoſed) the Numbers 109, 109, 109.

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
37	3, 7, 67
163	3, 7, 19, 67
29	13, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
4, 5	31
67	3, 7, 7, 31

Let us therefore now leave out 109, and conſequently the Squares of 263 and 499, where it is found; and ſee whether the

the remaining Squares will afford ſuch a Cube as is deſired. Now theſe are,

2	7	3	13	7	3, 19	29	13, 67	67	3, 7, 7, 31	191	7, 13, 13, 31
4	31	5	31	11	7, 19	37	3, 7, 67	163	3, 7, 19, 67	439	3, 31, 31, 67

7	3, 19
11	7, 19
163	3, 7, 19, 67
37	3, 7, 67
439	3, 31, 31, 67

7	3, 19
11	7, 19
163	3, 7, 19, 67
37	3, 7, 67
29	13, 67
3	13

7	3, 19
11	7, 19
163	3, 7, 19, 67
37	3, 7, 67
29	13, 67
191	7, 13, 13, 31
67	3, 7, 7, 31
4, 5	31

7	3, 19
11	7, 19
163	3, 7, 19, 67
29	13, 67
439	3, 31, 31, 67
191	7, 13, 13, 31

Of theſe, we will firſt begin with 19, which comes thrice (and but thrice) at the Squares of 7, 11, 163. Where we have 67 once, and therefore muſt have it twice more. Now if, for one of theſe, we take the Square of 37; we muſt, for the other, take either the Square of 439, or of 29. If that of 439; this brings in 3 a fourth time; which may not be, becauſe it comes not twice more to make up ſix times. Therefore (if at all) it muſt be that of 29, (or elſe 37 muſt be laid aſide;) But this brings in 13 once, for which we may have a ſecond at the Square of 3, but then we cannot have a third without a fourth, at the Square of 191. Therefore (waving that at the Square of 3) we muſt take both (if at all) at the Square of 191. Now this brings in 7 a fourth time, which calls for a fifth and ſixth: One of theſe we might have at the Square of 2; but then we cannot have a ſixth without a ſeventh. Therefore (waving that at 2) we muſt (if at all) take both at the Square of 67. But here, beſide a ſecond 31 (for which we may have a third at the Square of 4, or of 5,) we have 3 a fourth time (which will make up, not a Cube, but the Triple of a Cube,) which is not to be admitted, becauſe we cannot have a fifth and ſixth. And conſequently, the Square of 37 muſt be laid aſide, (as not to be joined either with that

that of 439 or 29;) but (waving that) we muſt have recourſe to the other two (at 29 and 439) for Tripling of 67. Now here we have 13 once; and therefore muſt have it twice more; not from the Square of 3, (becauſe, as before, if we take a ſecond here, we cannot have a third without a fourth;) but from that of 191. Which doth not only ſupply 13 twice; but alſo 7 and 31 which were alſo wanting: So that we have now a ſecond Cube, ſuch as was deſired; whoſe Components are, 3, 7, 13, 19, 31, 67, thrice taken. And the Square whence it ariſeth, is that of $7 \times 11 \times 29 \times 163 \times 191 \times 439$.

And if, from the remaining Square of 2, 4, 3, 5, 37, 67, we could form a third; this, Compounded with the laſt foregoing (as Prime to it) would form a fourth. But this cannot be, becauſe no Prime doth here thrice occur, but only 7 and 31: And neither of theſe can be thrice taken, without being incumbered with 3, which cannot be Tripled. So that, retaining 19 (as is hitherto ſuppoſed) we can have (from thence) no other Cube than what is already found.

2	7
4	31
3	13
5	31
37	3, 7, 67
67	3, 7, 7, 31

Let us now therefore lay by 19; and conſequently the Squares of 7, 11, 163, wherein it is found. And we have then theſe only left for conſideration.

2	7	3	13	29	13,67	67	3,7,7,31	439	3,31,31,67
4	31	5	31	37	3,7,67	191	7,13,13,31		

We have here 67 three times, at the Squares of 29, 37, 439. And (with theſe) we have 3 twice; which calls for a third from the Square of 67. And we have 13 once, for which we might have a ſecond at the Square of 3; but could not then have a third without a fourth; therefore (waving that) we take both from the Square of 191. And we have then 31 four times, and therefore muſt take it twice more from the Squares of 4, and of 5. But we have 7 four times;

29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31
191	7, 13, 13, 31
4	31
5	31

times; yet cannot find it twice more to make it up ſix times; nor indeed once more, becauſe we cannot here Compound the Square of 2, as being included in that of 4. So that, with 67, we may make up, not a Cube, but a Sextuple of a Cube.

Suppoſe we then that 67 be laid aſide; and therefore the Squares of 29, 37, 439. Thoſe that then remain are,

2|7 4|31 3|13 5|31 67|3,7,7,31 191|7,13,13,31

Of theſe, that of 67 muſt be laid aſide (becauſe 3 occurs but once,) and conſequently (becauſe 7 comes then but twice) that of 2 and 191. And for the other three (of 3, 4, 5,) the Number 13 comes but once; and 31 but twice. So that no further Cube can be hence expected.

4	31
5	31
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31
163	3, 7, 19, 67
191	7, 13, 13, 31
263	7, 7, 13, 109
439	3, 31, 31, 67
499	3, 7, 109, 109

We conclude therefore (having fully conſidered all) that (within the extent of this Table) we may have two Squares (and but two) ſuch as are deſired; whoſe Aggregate of Diviſors ſhall be a Cube. Namely, the Square of $7 \times 11 \times 29 \times 163 \times 191 \times 439$, whoſe Aggregate of Diviſors is the Cube of $3 \times 7 \times 13 \times 19 \times 31 \times 67$. And the Square of $4 \times 5 \times 7 \times 11 \times 37 \times 67 \times 163 \times 191 \times 263 \times 439 \times 499$; whoſe Aggregate of Diviſors is the Cube of $3 \times 3 \times 7 \times 7 \times 7 \times 13 \times 19 \times 31 \times 31 \times 67 \times 109$.

And, if any think it worth the pains to ſeek out more; they muſt enlarge the Table, to take in more Primes, or more Quadratick Powers of theſe Primes.

7	3, 19
11	7, 19
29	13, 67
163	3, 7, 19, 67
191	7, 13, 13, 31
439	3, 31, 31, 67

It had been eaſy to have rendered this buſineſs more ſtupendous (as ſome other would have done,) if (concealing the methods whereby I came at them) I would have performed the Multiplications here directed; and then, in thoſe great Numbers, exhi-

5 bited

bited thefe two Squares, with the two Cubes thence arifing; affirming, that (within fuch extent of Numbers) there is no other Square Number (befide thefe two, vaftly great,) which added to all its Aliquot Parts will make a Cube: Or perhaps, having affigned thofe two, propofed a Challenge to all the Mathematicians in *France*,) to find a third within thofe limits. But this would ferve only to amufe a Reader, not to inftruct him. And I chufe rather (in what I publifh) to inform my Reader, by what fteps I come at thofe difcoveries I make, and whereby he may (if he pleafe) attain the like; defigning more, the benefit of others, than oftentation.

I may here add (as is done after the former Queftion,) that the fame method is to be ufed, if (inftead of a Cube) it had been demanded, that fuch Aggregate fhould be the Triple (or other defigned Multiple) of a Cube: (fuppofing fuch defigned Multiple to be poffible:) Of which I have given fome inftances as I paffed along; and might have done more if it had been needful.

But we muft not then demand the Duple, Quadruple, Sextuple of a Cube, or otherwife Multiple thereof by an *even* Number: For all fuch are impoffible. For, fince every Quadratick power of a Prime Number (be it the firft, fecond, third, or further Square thereof,) hath, for its Divifors, (befide 1) all its Degrees or Powers fo far; (as, for inftance, a^6 hath for its Divifors 1, a, aa, a^3, a^4, a^5, a^6,) and all thefe (becaufe it is a Quadratick Power) are (excluding 1) in Number even; (and every of them either odd or even according as is the Prime a whence it arifeth;) and confequently, the Aggregate of all except 1, an even Number; (for an even Number of *odds*, as well as an even Number of *evens*, will ftill make an even Number;) to this even Number, if 1 be added (which is alfo an Aliquot Part, and therefore a Divifor,) this always makes the whole Aggregate an odd Number: Which therefore cannot be Duple of Cube, or its Multiple by an even Number. And the fame will hold as well for the Quadratick Powers of any Compound Number: For (as was fhewed before) the Aggregate of Divifors of fuch Compound Square, is always Compounded of fuch Aggregates of Divifors of fome Quadratick Powers of Primes; which, being (as is now fhewed) odd Numbers, their Compound muft be fo too. For an odd Number, Multiplied by an odd Number (and fo continually) will ftill produce an odd Number; and therefore, not the Duple (or otherwife Multiple by an even Number) of any Number whatfoever.

In the former Queftion, concerning Cubick Powers, whofe Aggregate of Divifors fhould be equal to a Square, (or a defigned Multiple of a Square,) this will not hold, For there the Aggregate may be either an odd or an even Number. Yet with this diverfity: If the Prime *a* be 2, then all the Degrees thereof will be even Numbers, to which when 1 is added the Aggregate will be odd. If the Prime *a* be 3 (or other odd Prime,) and the Cube thence arifing be the firft, third, fifth Cube, (or other in odd places) whofe Number of dimenfions is 3, 9, 15, or other odd Number; the Number of Divifors, without 1, will be odd alfo; and therefore, with 1, it will become even. But if fuch Prime *a*, be odd, and the Cubick Power thereof be the fecond, fourth, fixth, or other in even places, whofe Number of dimenfions will therefore be 6, 12, 18, or other even Number (which will therefore be Quadratick as well as Cubick;) here the Number of Divifors without 1, will be even, and their Aggregate even; and therefore with 1, the Aggregate will be odd. And accordingly an eftimate is to be made of the Compounds of fuch Aggregates: For, if all the Compounding Aggregates be odd, the Compound will be alfo odd; but if any one of them be even, the Compound Aggregate will be even. I forbear to purfue this to any nicer determination: But any who pleafe may purfue it further.

Of

Of the Third Queſtion mentioned above in pages 322, 323; *which was propoſed by Dr.* WALLIS *to Monſieur* FERMAT.

III. A third Queſtion I added to thoſe two; not as a new difficulty, but as a trial whether Monſieur *Fermat* did thoroughly underſtand the myſtery of his own two Queſtions; and did not only by chance light on them: For if he thoroughly underſtood thoſe, he muſt needs be able to ſolve this with much eaſe; which it ſeems, by Epiſt. 37, he did not find ſo eaſy; and therefore, what ſolution he did find, he choſe rather to conceal than let us know it. Nor doth any where let us know, whether he were able to ſolve his own Queſtions. But Monſieur *Frenicle* gives ſolutions both of this and thoſe; but without acquainting us by what methods he came at them; which makes me think they are not better than mine.

The Queſtion is this: *To find two Square Numbers, which added to their Aliquot Parts ſhall make the ſame Number* (or, whoſe Aggregate of Diviſors ſhall be the ſame;) *As for inſtance* $16 + 8 + 4 + 2 + 1 = 31 = 25 + 5 + 1$; *Let two ſuch other be found.*

Now 'tis manifeſt (by what hath been before delivered) that any Multiple of thoſe two (16 and 25) by any other Square which is a Prime to both of them (as 9, 49, 121, &c,) will do what is deſired. For the Multiple of 31, by the Aggregate of Diviſors of any ſuch other Square, will be the Aggregate of Diviſors, both of 16, and of 25, Multiplied by ſuch Square. As for inſtance, becauſe $9 + 3 + 1 = 13$; therefore $31 \times 13 = 403$, is the Aggregate of the Diviſors, as well of $16 \times 9 = 144$, as of $25 \times 9 = 225$.

But, if we would have others than the Equimultiples of 16 and 25; we may make uſe of the former Table of Squares; wherein (becauſe we do not meet with any ſingle Squares,

Squares, (other than thoſe of 4 and of 5,) whoſe Aggregate of Diviſors is the ſame) we are ſo to Compound two or more of them in ſeveral parties, as that the Aggregates be the ſame. As, the Squares of

$$\left.\begin{array}{l}4\\5\end{array}\right\}31. \qquad \left.\begin{array}{l}29\times67\\2\times3\times5\times37\end{array}\right\}3\times7\times7\times13\times31\times67.$$

$$\left.\begin{array}{l}2\times19\times29\\3\times8\times37\end{array}\right\}3\times7\times13\times67\times127.$$

$$\left.\begin{array}{l}7\times8\times29\times67\\3\times4\times11\times19\times37\end{array}\right\}3\times3\times7\times7\times13\times19\times31\times67\times127.$$

$$\left.\begin{array}{l}7\times8\times92\times76\\3\times5\times11\times19\times37\end{array}\right\}3\times3\times7\times7\times13\times19\times31\times67\times127.$$

All which ariſe from Compounding the Squares of the Primes leſs than 100, taking into the Number the ſecond and third Squares of 2.

And more Couples than theſe are not to be found within thoſe limits, unleſs by Multiplying both the Numbers of ſome of theſe Couples by ſome common Square which is a Prime to both of them; which may be done at pleaſure. But if we extend the limits, to other Primes, and other Powers of theſe Primes, we may have more without ſtint.

And by the ſame means we may have Three or more ſuch Squares, whoſe Aggregate of Diviſors ſhall make the ſame ſum. As (amongſt theſe) we have Three. Namely the Squares of

$$\left.\begin{array}{l}7\times8\times29\times67\\3\times4\times11\times19\times37\\3\times5\times11\times19\times37\end{array}\right\}3\times3\times7\times7\times13\times19\times31\times67\times127.$$

But if we enlarge the bounds, we may find others (Two's, Threes, Fours, &c,) in great Multitudes, whoſe Aggregate of of Divifors fhall be the fame. As any man by experience, may find, who (without going farther) will give himfelf the trouble of purfuing the whole Table here given, as I have done thofe Primes which are fmaller than 100.

I forbear to purfue more Queftions of this nature; but, according to the fame method, any others of like kind may be difpatched.

F I N I S.

AN

APPENDIX

TO THE

ENGLISH TRANSLATION

OF

RHONIUS'S GERMAN TREATISE OF ALGEBRA,

MADE BY

MR. THOMAS BRANCKER, M. A.

And Publiſhed by him,

With the Advice and Aſſiſtance of Dr. JOHN PELL,

At London, in the Year 1668;

CONTAINING

A TABLE OF ODD NUMBERS

LESS THAN ONE HUNDRED THOUSAND,

SHEWING,

Firſt, Which of them are INCOMPOSIT, or PRIME, NUMBERS,

And, Secondly, The FACTORS, or CO-EFFICIENTS, by the Multiplication of which the others are produced; Supputated, or Computed, by the ſame THOMAS BRANCKER.

THE

TRANSLATOR'S PREFACE.

THE *Title-Page ſays that this Book was a* Tranſlation, *but hath been much* altered. *If any man deſire to know* what *the* alterations *are, and* why *they were made; he may do well to compare it with the Original: A Printed Copy whereof may be had at* Francfort *in* Germany, *by any that inquires there for it by this Title,* Algebra Rhonii Germanicè; Tiguri * apud Bodmerum, 1659, in quarto. *The Copy which I have, was given me* anno 1662, *by a good Friend, who* then *told me he much deſired to read it in ſome Language that he underſtood; I then promiſed him to* Engliſh *it. As ſoon as my leiſure permitted, I corrected it according to the Printed Catalogue of Errata, and then began the Tranſlation. When it was finiſhed, I deſired to ſee it Printed, and got it Licenſed* May 18, 1665, *with the name of* An Introduction to Algebra. *And ſo* without any alteration *either in the Precepts or Examples, ſave only the correction of many Miſtakes: It was ſent to the Preſs, with order to Re-print the ſix leaves of* His Table of Incompoſits *preciſely as they ſtand there.*

A little after, I heard that there was at that time in London, *a Perſon of Note* † *very worthy to be made acquainted with my deſign, before I made any farther progreſs in the Impreſſion. Being admitted to ſpeak with him, I found him not only* able *to direct me, but alſo very* willing *ſo to do, ſo far as his leiſure would permit. He gave me divers cautions concerning the Work. He ſhewed me the way of making the* Table of Incompoſits, *of examining it, and of continuing it as far as I would. He encouraged me to extend it to* 100 *thouſand: Telling me that by that time that I had Calculated and Printed that Table, he*

* That is, at Zurich, in Switzerland.

† Dr. John Pell.

hoped to be at leiſure to review ſome of Monſieur Rhonius *his Problemes, and to work them anew; and that he would ſend them to me, with leave to publiſh them or to keep them by me.*

I had finiſhed and Printed that Table, as alſo Twelve ſheets of the Book itſelf, *before he ſent me his* Alterations. *They begin with* Probl. 24, pag. 100. *All from thence to the end is his Work: As alſo* pag. 79, 80, 81, 82, *which he ſent laſt of all: So that inſtead of the firſt* 124 pages *of* Rhonius, this *hath juſt twice as many: Inſtead of thoſe* 8 *or* 9 *ſheets remaining in* Rhonius, *how much ſhall be hereafter publiſhed, I will not adventure to foretell, becauſe of the uncertainty of life, health, leiſure, and of the acceptance which* this *ſhall find amongſt the Lovers of theſe Studies, to whom this might have been more acceptable, if it had been wholly void of Preſs-faults.*

As for the Table of Incompoſits, *I was very ſenſible of the bad effects of perfunctorineſs in Supputating, Tranſcribing, or Printing of it. My care therefore was not ſmall: yet* pag. 198, *is almoſt filled with* Errata, *and I dare not warrant that none have eſcaped unſeen: But ſeeing ſo few are fit to undertake to Supputate it anew, whoſoever ſhall happen to diſcover any other fault in that Table, ſhall do well to ſignify it to the* Book-ſeller, *or to any other likely to be concerned in the next Impreſſion.*

The Errata *in* the reſt of the Book *are many, notwithſtanding my care, and the diligence of a good friend, who Corrected part of it, after my removal to an abode ſo far from* London. *Moſt of them cannot trouble the more exerciſed ſort of Readers. But fear of leaving any ſtumbling-block in the way of Beginners hath cauſed this larger Enumeration of them in the three next following* pages.

White-gate *in* Cheſhire,
April 22, 1668. T. B.

From

From pages 34 and 35 of BRANCKER'*s Tranſlation of* RHONIUS'*s Algelra.*

BUT it is oftentimes very troubleſome to find a Square, Cube, &c, whereby this *Abbreviation* may be performed. Find therefore all the *Partes aliquotæ*, or *juſt Dividers*, and theſe will tell us whether, and how often any Cube, Square, &c, is contained in the Quantity aſſigned.

Foraſmuch then as the Diſcovery of the *Partes aliquotæ* is many waies uſeful in *Vulgar Arithmetick*, I have adjoyned a Table in the End of this Book, which diſcovers them in all uneven Numbers as far as 100,000.

In which Table [p] ſtands for a *Prime Number* throughout.

The Uſe of that Table is

To diſcover *at view* whether any given Quantity be compound or ſimple, *i. e.* be diviſible or indiviſible, and how many *Partes aliquotæ* it hath. On the left ſide you ſee, run down all the odd Numbers to 99, which muſt be ſet after the Numbers in the Head-Row, as Occaſion is, thus. Let the Number given be 21449, ſeek 49 in the ſide, and the other 214 in the head, then run downward, and ſide-waies till their Rows meet in a Square, where we find 89, which is a *Pars aliquota*, which dividing 21449, gives Quotient 241. With this 241 do as before (*i. e.* ſeek 41 on the ſide, and 2 in the head) and in its Square you find (P) which ſhews that it is an indiviſible or *Prime* Number. Wherefore the aliquot Parts of this 21449 ſtand thus.

1
89 · 241
21449.

If

If the even Number 21696 were given, *ſubdivide* it continually by 2 till the Quotient be an odd Number (as at the ſixth Time you will here find 339.) Seek this 339 in the Table as you are directed above. In its Square we find 3, which dividing 339 gives Quotient 113, which 113 we find to be a *Prime* Number. The *Partes aliquotæ* of the Number 21696 ſtand as follows. Out of 1, 2, 3, 113, we may find the reſt.

1

2 . 2 . 2 2 . 2 . 2

4 . 8 . 16 . 32 . 64

3 . 6 . 12 . 24 . 48 . 96 . 192

113 . 226 . 452 . 904 . 1808 . 3616 . 7232 . 339 . 678 . 1356 . 2712 . 5424
10848 . 21696.

How thoſe *Principal* Diviſors (1, 2, 3, 113,) are multiplied into each other, and into their Products, lies plain before the Eyes without any more words.

Mr. THOMAS BRANCKER*'s Preface to his long Table of Incompoſit, or Prime, Numbers; from pages* 193, 194, 195, 196, 197, *and* 198, *of his Tranſlation of* RHONIUS*'s Algebra, publiſhed in the Year* 1668.

This is *the Table* mentioned page 34, line 8. It fills 50 pages. Its firſt page calls it *a Table of Incompoſit numbers leſs than* 100,000; but it contains far more *compoſit* numbers, than *incompoſit*; For it doth not only give an *Orderly enumeration of all odd numbers* which are not *compoſit*: but alſo it ſhews that none of the reſt are ſo. To every other odd number there expreſſed, the Table ſets ſome *incompoſit* that will *divide it without fraction.*

Each page hath 21 columels, whereof the *firſt* is filled with 40 odd numbers ſtanding in their natural order. The *following twenty* columels are diſtinguiſhed on their Tops, by numbers

numbers in their natural order 0, 1, 2, 3, to 998, 999. Theſe *Top numbers* are *hundreds*; the 40 *marginal* numbers are Unites *adhering to the Centuries*. A line running from any marginal croſs the page, ſhews, in any column, the *place* of the number made up of the Top-number and that marginal. In every ſuch *place* of concourſe you ſhall either find the letter *p*, or ſome *incompoſit* leſs than 317. The letter *p* ſhews the number to be a *prime or incompoſit*, (See Euclid, VII. def. 11 and 13.) If any number leſs than 100,000, do end in 1, 3, 7, or 9, you may find its place in one of thoſe 50 pages, and then ſee whether it be a *prime* or no: If it be compoſit, you will there find its *leaſt Diviſor*. Thus in page 1, where the line marked with the marginal 67, croſſeth the columel whoſe Top-number is 16; there you find *p*, that is, 1667 is a prime. Where the ſame line croſſeth the next columel, you find 3; That is, 1767 is no prime, and 3 is the *leaſt Diviſor of it*. So in page 25, you ſee 49031, 49033, 49037 are primes; but 49039 is a Compoſit, and 19 is its ſmalleſt Diviſor.

It may be of great uſe ſometimes to have *a complete and orderly enumeration of all incompoſits between* 0, *and* 100,000, *without any mixture of Compoſits*; thus 1. 2. 3. 5. 7. 11. 13. &c, leaving out 9, 21 and all other compoſits. The numbers 2 and 5 are primes, though they be left out of the long Table, becauſe no other incompoſit ends ſo. Theſe two prime numbers 2 and 5 being duely placed, all the reſt of the primes are taken out of the long Table as they there ſtand *marked with p*, from 7 in the firſt page to 99991 in the end of the 50*th* page.

If to *each of theſe primes* you ſet the Briggian Logarithm, you may find the Logarithms for *all the reſt of the numbers* in the firſt 100 Chiliads, by addition of the Logarithms of their incompoſit Factors.

The Reſolving of a number into all its incompoſit Factors [as 4620 into 2. 2. 3. 5. 7. 11.] is altogether neceſſary, for the determining *how many Diviſors* that number hath, and *which they be*: As in pages 194, 195.

(29)
1.abcdef
a. bcdef
b. acdef
c. abdef
d. abcef
e. abcdf
f. abcde
ab. cdef
ac. bdef
ad. bcef
ae. bcdf
af. bcde
bc. adef
bd. acef
be. acdf
bf. acde
cd. abef
ce. abdf
cf. abde
de. abcf
df. abce
ef. abcd
abc. def
abd. cef
abe. cdf
abf. cde
acd. bef
ace. bdf
acf. bde
ade. bcf
adf. bce
aef. bcd

(28)
1.aabcde
a. abcde
b. aacde
c. aabde
d. aabce
e. aabcd
aa. bcde
ab. acde
ac. abde
ad. abce
ae. abcd
bc. aade
bd. aace
be. aacd
cd. aabe
ce. aabd
de. aabc
aab. cde
aac. bde
aad. bce
aae. bcd
abc. ade
abd. ace
abe. acd

(27)
1.aabbcd
a. abbcd
b. aabcd
c. aabbd
d. aabbc
aa. bbcd
ab. abcd
ac. abbd
ad. abbc
bb. aacd
bc. aabd
bd. aabc
cd. aabb
aab. bcd
aac. bbd
aad. bbc
abb. acd
abc. abd

(26)
1.aaabcd
a. aabcd
b. aaacd
c. aaabd
d. aaabc
aa. abcd
ab. aacd
ac. aabd
ad. aabc
bc. aaad
bd. aaac
cd. aaab
aaa. bcd
aab. acd
aac. abd
aad. abc

(25)
1.aabbcc
a. abbcc
b. aabcc
c. aabbc
aa. bbcc
ab. abcc
ac. abbc
bb. aacc
bc. aabc
cc. aabb
aab. bcc
aac. bbc
abb. acc
abc. abc

(24)
1.aaabbc
a. aabbc
b. aaabc
c. aaabb
aa. abbc
ab. aabc
ac. aabb
bb. aaac
bc. aaab
aaa. bbc
aab. abc
aac. abb

(23)
1.aaaabc
a. aaabc
b. aaaac
c. aaaab
aa. aabc
ab. aaac
ac. aaab
bc. aaaa
aaa. abc
aab. aac

(22)
1.aaabbb
a. aabbb
b. aaabb
aa. abbb
ab. aabb
bb. aabb
aaa. bbb
aab. abb

(21)
1.aaaabb
a. aaabb
b. aaaab
aa. aabb
ab. aaab
bb. aaaa
aaa. abb
aab. aab

(20)
1.aaaaab
a. aaaab
b. aaaaa
aa. aaab
ab. aaaa
aaa. aab

(19)
1.aaaaaa
a. aaaaa
aa. aaaa
aaa. aaa

(18)
1. abcde
a. bcde
b. acde
c. abde
d. abce
e. abcd
ab. cde
ac. bde
ad. bce
ae. bcd
bc. ade
bd. ace
be. acd
cd. abe
ce. abd
de. abc

(17)
1. aabcd
a. abcd
b. aacd
c. aabd
d. aabc
aa. bcd
ab. acd
ac. abd
ad. abc
bc. aad
bd. aac
cd. aab

(16)
1. aabbc
a. abbc
b. aabc
c. aabb
aa. bbc
ab. abc
ac. abb
bb. aac
bc. aab

(15)
1.aaabc
a. aabc
b. aaac
c. aaab
aa. abc
ab. aac
ac. aab
bc. aaa

(14)
1.aaabb
a. aabb
b. aaab
aa. abb
ab. aab
bb. aaa

(13)
1.aaaab
a. aaab
b. aaaa
aa. aab
ab. aaa

(12)
1.aaaaa
a. aaaa
aa. aaa

(11)
1. abcd
a. bcd
b. acd
c. abd
d. abc
ab. cd
ac. bd
ad. bc

(10)
1. aabc
a. abc
b. aac
c. aab
aa. bc
ab. ac

(9)
1.aabb
a. abb
b. aab
aa. bb
ab. ab

(8)
1.aaab
a. aab
b. aaa
aa. ab

(7)
1.aaaa
a. aaa
aa. aa

(6)
1. abc
a. bc
b. ac
c. ab

(5)
1.aab
a. ab
b. aa

(4)
1.aaa
a. aa

(3)
1. ab
a. b

(2)
1. aa
a. a

(1)
1. a

(29)
1.30030
2.15015
3.10010
5.6006
7.4290
11.2730
13.2310
6.5005
10.3003
14.2145
22.1365
26.1155
15.2002
21.1430
33. 910
39. 770
35. 858
55. 546
65. 462
77. 390
91. 330
143.210
30.1001
42.715
66.455
78.385
70.429
110.273
130.231
154 195
182.165
286.105

(28)
1.4620
2.2310
3.1540
5.924
7.660
11.420
4.1155

6.770
10.462
14.330
22.210
15.308
21.220
33.140
35.132
55. 84
77. 60
12.385
20.231
28.165
44.105
30.154
42.110
66. 70

(27)
1.1260
2.630
3.420
5.252
7.180
4.315
6.210
10.126
14. 90
9. 140
15. 84
21. 60
35. 36
12.105
20. 63
28. 45
18. 70
30. 42

(26)
1.840
2.420
3.280

5.168
7.120
4.210
6.140
10. 84
14. 60
15. 56
21. 40
35. 24
8. 105
12. 70
20. 42
28. 30

(25)
1.900
2.450
3.300
5.180
4.225
6.150
10. 90
9.100
15. 60
25. 36
12. 75
20. 45
18. 50
30. 30

(24)
1.360
2.180
3.120
5. 72
4. 90
6. 60
10. 36
9. 40
15. 24
8. 45
12. 30

20.18

(23)
1.240
2.120
3. 80
5. 48
4. 60
6.40
10.24
15.16
8.30
12.20

(22)
1.216
2.108
3.72
4.54
6.36
9.24
8.27
12.18

(21)
1.144
2.72
3.48
4.36
6.24
9.16
8.18
12.12

(20)
1.96
2.48
3.32
4.24
6.16
8.12

(19)
1.64
2.32

4. 16
8. 8

(18)
1.2310
2.1155
3. 770
5. 462
7. 330
11.210
6.385
10.231
14.165
22.105
15.154
21.110
33. 70
35. 66
55. 42
77. 30

(17)
1.420
2.210
3.140
5. 84
7. 60
4.105
6. 70
10. 42
14. 30
15. 28
21. 20
35. 12

(16)
1.180
2. 90
3. 60
5. 36
4. 45
6. 30
10. 18

9.20
15.12

(15)
1.120
2. 60
3. 40
5. 24
4. 30
6. 20
10.12
15. 8

(14)
1.72
2.36
3.24
4.18
6.12
9. 8

(13)
1.48
2.24
3.16
4.12
6. 8

(12)
1.32
2.16
4. 8

(11)
1.210
2.105
3.70
5.42
7.30
6.35
10.21
14.15

(10)
1.60

2.30
3.20
5.12
4.15
6.10

(9)
1.36
2.18
3.12
4. 9
6. 6

(8)
1.24
2.12
3. 8
4. 6

(7)
1.16
2. 8
4. 4

(6)
1.30
2.15
3.10
5. 6

(5)
1.12
2. 6
3. 4

(4)
1. 8
2. 4

(3)
1. 6
2. 3

(2)
1. 4
2. 2

(1)
1. 2

Sort	Forme	Div.
29	*abcdef*	64
28	*aabcde*	48
27	*aabbcd*	36
26	*aaabcd*	32
25	*aabbcc*	27
24	*aaabbc*	24
23	*aaaabc*	20
22	*aabbb*	16
21	*aaabb*	15
20	*aaaaab*	12
19	*aaaaaa*	7
18	*abcde*	32
17	*aabcd*	24
16	*aabbc*	18
15	*aaabc*	16
14	*aaabb*	12
13	*aaaab*	10
12	*aaaaa*	6
11	*abcd*	16
10	*aabc*	12
9	*aabb*	9
8	*aaab*	8
7	*aaaa*	5
6	*abc*	8
5	*aab*	6
4	*aaa*	4
3	*ab*	4
2	*aa*	3
1	*a*	2

That is, the 29th sort hath 64 Divisors; the 18th hath but 32, &c.

Use of the Long Table of Numbers, ending in 1, 3, 7, *or* 9.

Every *Aliquot part* of a Number is one of the juſt Diviſors of it. The greateſt Diviſor being equal to the *whole* Dividend, muſt not be called a *Part:* Wherefore, ſubſtract 1 from every number in the laſt columel of page 195, you ſhall have the number of *aliquot* parts belonging to every one of thoſe 29 ſorts.

Having the leaſt Diviſor of any Number of the long Table, to find all its other incompoſit Co-efficients.

If that Diviſor end in 1 or 9, and have a black ſtroke *under* it in the Dividend's place in the long table; or if the Diviſor end in 3 or 7, and have ſuch a ſtroke *over* it in the Dividend's place; the Dividend is the *ſquare of an incompoſit*, and the Quotient is given, for it is equal to the Diviſor.

If the leaſt Diviſor have no ſuch ſtroke by it, let it divide the propoſed number, the Quotient ſhall be the greateſt *aliquot part* of that Dividend: Seek that Quotient in the ſame long Table; if it be there *marked with p*, your inquiry is at an end; the Dividend is of the form AB. If it be *not ſo marked*, by the Prime there found, divide your *firſt* Quotient, deal with the *ſecond* Quotient as you had done with the *firſt*, repeating ſuch Diviſions, till the Quotient be incompoſit. Thus 53191 is found in page 27, with its ſmalleſt

ſmalleſt Diviſor 43. Now 53191 divided by 43 gives 1237. Page 1 ſays, this 1237 is a prime. Inquire no farther.

But, deſiring the incompoſit factors of 93611, I find it in page 47 of the long Table, with 7 for its leaſt Diviſor. The Quotient 13373 is found in page 7, with its leaſt Diviſor 43. This 43 gives a ſecond quotient 311. Page 1 ſays, this 311 is an incompoſit. So the prime Co-efficients of 93611 are 7. 43. 311. (Hence infer that 53191 is to 93611, as 1237 to 2177 = 7 * 311, or 7 × 311.

If you divide *any odd* number by *all the primes in order*, beginning with 3, The firſt Diviſor that finds a Quotient without fraction, is the leaſt Diviſor that the Dividend can have. Thus, 239 is the leaſt number that meaſures 1 111 111. Try 3, 7, 11, &c. No prime can divide 1 111 111 till you come to 239. If no ſuch Diviſor find an Integer Quotient, before the Quotient is leſs than the Diviſor, pronounce your Dividend to be incompoſit, and that laſt Diviſor to be greater than the Dividend's ſquare root. Frequent occaſion of Dividing by Incompoſits calls for a *Tariffa* of as many *primes* as ſhall be needful. For reſolving of numbers leſs than 100,000, it ſufficeth if it be extended to 313, as in the next page.

A Tariffa, or Table, of all Incomposit, or Prime, Numbers, less than √ 100,000, *multiplied by* 2, 3, 4, 5, 6, 7, 8, 9.

1	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73
2	4	6	10	14	22	26	34	38	46	58	62	74	82	86	94	106	118	122	134	142	146
3	6	9	15	21	33	39	51	57	69	87	93	111	123	129	141	159	177	183	201	213	219
4	8	12	20	28	44	52	68	76	92	116	124	148	164	172	188	212	236	244	268	284	292
5	10	15	25	35	55	65	85	95	115	145	155	185	205	215	235	265	295	305	335	355	365
6	12	18	30	42	66	78	102	114	138	172	186	222	246	258	282	318	354	366	402	426	438
7	14	21	35	49	77	91	119	133	161	203	217	259	287	301	329	371	413	427	469	497	511
8	16	24	40	56	88	104	136	152	184	232	248	296	328	344	376	424	472	488	536	568	584
9	18	27	45	63	99	117	153	171	207	261	279	333	369	387	423	477	531	549	603	939	657

1	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151	157
2	158	166	178	194	202	206	214	218	226	254	262	274	278	298	302	314
3	237	249	267	291	303	309	321	327	339	381	393	411	417	447	453	471
4	316	332	356	388	404	412	428	436	452	508	524	548	556	596	604	628
5	395	415	445	485	505	515	535	545	565	635	655	685	695	745	755	785
6	474	498	534	582	606	618	642	654	678	762	786	822	834	894	906	942
7	553	581	623	679	707	721	749	763	791	889	917	959	973	1043	1057	1099
8	632	664	712	776	808	824	856	872	904	1016	1048	1096	1112	1192	1208	1256
9	711	747	801	873	909	927	963	981	1017	1143	1179	1233	1251	1341	1359	1413

1	163	167	173	179	181	191	193	197	199	211	223	227	229	233
2	326	334	346	358	362	382	386	394	398	422	446	454	458	466
3	489	501	519	537	543	573	579	591	597	633	669	681	687	699
4	652	668	692	716	724	764	772	788	796	844	892	908	916	932
5	815	835	865	895	905	955	965	985	995	1055	1115	1135	1145	1165
6	978	1002	1038	1074	1086	1146	1158	1182	1194	1266	1338	1362	1374	1398
7	1141	1169	1211	1253	1267	1337	1351	1379	1393	1477	1561	1589	1603	1631
8	1304	1336	1384	1432	1448	1528	1544	1576	1592	1688	1784	1816	1832	1864
9	1467	1503	1557	1611	1629	1719	1737	1773	1791	1899	2007	2043	2061	2097

1	239	241	251	257	263	269	271	277	281	283	293	307	311	313
2	478	482	502	514	526	538	542	554	562	566	586	614	622	626
3	717	723	753	771	789	807	813	831	843	849	879	921	933	939
4	956	964	1004	1028	1052	1076	1084	1108	1124	1132	1172	1228	1244	1252
5	1195	1205	1255	1285	1315	1345	1355	1385	1405	1415	1465	1535	1555	1565
6	1434	1446	1506	1542	1578	1614	1626	1662	1686	1698	1758	1842	1866	1878
7	1673	1687	1757	1799	1841	1883	1897	1939	1967	1981	2051	214[illegible]	2177	2191
8	1912	1928	2008	2056	2104	2152	2168	2216	2248	2264	2344	2456	2488	2504
9	2151	2169	2259	2313	2367	2421	2439	2493	2529	2547	2637	2763	2799	2817

Amendments to the following Table.

P. Guldin sayes, 149 is divisible by 7, and 229 by 31. *Schooten* leaves 809 out of his *Catalogue of Incomposits. Rhonius* makes 1209 and 1673 incomposits, and says 11833 is divisible by 19. But *this Table* sayes more truly, that 149.229. 809.11833 are Incomposits: and that 1209 is divisible by 3, and 1673 by 7. Yet trust it not, before you have amended *these faults* in it.

Pa	Numb.	For.	Set.	Pa	Numb.	For.	Set.	Pa	Numb.	For.	Set.	Pa	Numb.	For.	Set.
5	9211	19	61	21	40277	13	p		60779	63	163		72381	p	3
	9799	40	41		40591	3	p		61779	p	3		72383	3	p
6	10199	p	7		40593	p	3	32	62011	3	p		72557	73	37
	10813	13	11		40597	3	p		62013	p	3		72601	97	79
7	13201	23	43	23	44659	11	17		62017	3	p		73023	p	3
9	17563	3	7		45353	p	7		62019	p	3		73051	7	11
	17981	41	p		45837	p	3		63839	71	p		73481	179	197
10	18903	7	3	24	466 . .	476	466		63883	191	193		73493	p	7
	18907	3	7		46089	7	3	33	641 . .	541	641		73913	p	7
	18909	7	3		46457	3	p		659 . .	569	659	40	78199	p	11
12	23203	3	p		47201	11	7		64237	61	p	41	80333	67	11
	23381	193	103		47577	7	3		64693	3	p		80663	p	11
13	24011	3	13		47579	3	7		64973	23	43	42	83123	103	101
	25093	13	23		47663	p	7		65959	17	71	43	84311	57	59
	25873	23	p	25	48601	53	7	34	66239	19	p	44	86699	281	181
14	27233	31	113	27	53361	7	3		66561	7	3	46	91189	p	7
	27517	p	7		53791	3	p		66563	3	7		91707	p	3
15	28201	3	p	28	54507	7	3		66567	p	3		91793	23	17
	28203	p	3		54509	3	7		66569	3	p	47	92701	3	7
	29599	blank	p		54589	71	79		66761	191	101		92703	7	3
17	32297	71	p	29	56323	157	151		66951	2	3		92773	163	113
	33259	97	79	30	58123	11	13	35	68809	53	13		93101	151	157
	33591	7	3		58181	71	73	36	70313	167	p		93161	52	59
	33593	3	7		58301	137	173		70981	167	p	48	95371	281	283
18	34089	7	3		58901	blank	p		71113	3	7		95797	p	13
	34209	23	3		59901	7	3		71603	p	7	49	97903	3	13
	35089	3	p		59909	137	139		71983	167	p	50	98099	26	263
20	39263	p	7	31	60079	63	73	37	72357	7	3		98551	39	139
	39589	p	11		60293	7	p		72359	3	7		99443	17	277

THOMAS

Mr. THOMAS BRANCKER'*s* TABLE *of* INCOMPOSIT, *or* PRIME, NUMBERS, *less than* 100,000.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
01	1. P	P	3	7	P	3	P	P	3	17	7	3	P	P	3	19	P	3	P	P
03	P	P	7	3	13	P	3	19	11	3	17	P	3	P	23	3	7	13	3	11
07	P	P	3	P	11	3	P	7	3	P	19	3	17	P	3	11	P	3	13	P
09	3	P	11	3	P	P	3	P	P	3	P	P	3	7	P	3	P	P	3	23
11	P	3	P	P	3	7	13	3	P	P	3	11	7	3	17	P	3	29	P	3
13	P	P	3	P	7	3	P	23	3	11	P	3	P	13	3	17	P	3	7	P
17	P	3	7	P	3	11	P	3	19	7	3	P	P	3	13	37	3	17	23	3
19	P	7	3	11	P	3	P	P	3	P	P	3	23	P	3	7	P	3	17	19
21	3	11	13	3	P	P	3	7	P	3	P	19	3	P	7	3	P	P	3	17
23	P	3	P	17	3	P	7	3	P	13	3	P	P	3	P	P	3	P	P	3
27	3	P	P	3	7	17	3	P	P	3	13	7	3	P	P	3	P	11	3	41
29	P	3	P	7	3	23	17	3	P	P	3	P	P	3	P	11	3	7	31	3
31	P	P	3	P	P	3	P	17	3	7	P	3	P	11	3	P	7	3	P	P
33	3	7	P	3	P	13	3	P	7	3	P	11	3	31	P	3	23	P	3	P
27	P	P	3	P	19	3	7	11	3	P	17	3	P	7	3	29	P	3	11	13
39	3	P	P	3	P	7	3	P	P	3	P	17	3	13	P	3	11	37	3	7
41	P	3	P	11	3	P	P	3	29	P	3	7	17	3	11	23	3	P	7	3
43	P	11	3	7	P	3	P	P	3	23	7	3	11	17	3	P	31	3	19	29
47	P	3	13	P	3	P	P	3	7	P	3	31	29	3	P	7	3	P	P	3
49	7	P	3	P	P	3	11	7	3	13	P	3	P	19	3	P	17	3	43	P
51	3	P	P	3	11	19	3	P	23	3	P	P	3	7	P	3	13	17	3	P
53	P	3	11	P	3	7	P	3	P	P	3	P	7	3	P	P	3	P	17	3
57	3	P	P	3	P	P	3	P	P	3	7	13	3	23	31	3	P	7	3	19
59	P	3	7	P	3	13	P	3	P	7	3	19	P	3	P	P	3	P	11	3
61	P	7	3	19	P	3	P	P	3	31	P	3	13	P	3	7	11	3	P	37
63	3	P	P	3	P	P	3	7	P	3	P	P	3	29	7	3	P	41	3	13
67	P	P	3	P	P	3	23	13	3	P	11	3	7	P	3	P	P	3	P	7
69	3	13	P	3	7	P	3	P	11	3	P	7	3	37	13	3	P	29	3	11
71	P	3	P	7	3	P	11	3	13	P	3	P	31	3	P	P	3	7	P	3
73	P	P	3	P	11	3	P	P	3	7	29	3	19	P	3	11	7	3	P	P
77	7	3	P	13	3	P	P	3	P	P	3	11	P	3	7	19	3	P	P	3
79	P	P	3	P	P	3	7	19	3	11	13	3	P	7	3	P	23	3	P	P
81	3	P	P	3	13	7	3	11	P	3	23	P	3	P	P	3	41	13	3	7
83	P	3	P	P	3	11	P	3	P	P	3	7	P	3	P	P	3	P	7	3
87	3	11	7	3	P	P	3	P	P	3	P	P	3	19	P	3	7	P	3	P
89	P	3	17	P	3	19	13	3	7	23	3	29	P	3	P	7	3	P	P	3
91	7	P	3	17	P	3	P	7	3	P	P	3	P	13	3	37	19	3	31	11
93	3	P	P	3	17	P	3	13	19	3	P	P	3	7	P	3	P	11	3	P
97	P	P	3	P	7	3	17	P	3	P	P	3	P	11	3	P	P	3	7	P
99	3	P	13	3	P	P	3	17	29	3	7	11	3	P	P	3	P	7	3	P

	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
01	3	11	31	3	7	41	3	37	p	3	p	7	3	p	19	3	13	p	3	47
03	p	3	p	7	3	p	19	3	p	p	3	29	p	3	41	31	3	7	p	3
07	3	7	p	3	29	23	3	p	7	3	31	13	3	p	p	3	p	11	3	p
09	7	3	47	p	3	13	p	3	53	p	3	p	p	3	7	11	3	p	13	3
11	p	p	3	p	p	3	7	p	3	41	p	3	13	7	3	p	23	3	37	p
13	3	p	p	3	19	7	3	p	29	3	23	11	3	p	p	3	p	47	3	7
17	p	29	3	7	p	3	p	11	3	p	7	3	p	31	3	p	p	3	11	p
19	3	13	7	3	41	11	3	p	p	3	p	p	3	p	13	3	7	p	3	p
21	43	3	p	11	3	p	p	3	7	23	3	p	p	3	11	7	3	61	p	3
23	7	11	3	23	p	3	43	7	3	37	p	3	11	p	3	13	p	3	p	p
27	p	3	17	13	3	7	37	3	11	p	3	53	7	3	23	p	3	p	43	3
29	p	p	3	17	7	3	11	p	3	29	13	3	p	p	3	p	19	3	7	p
31	3	p	23	3	11	p	3	p	19	3	7	31	3	p	47	3	p	7	3	p
33	19	3	7	p	3	17	p	3	p	7	3	13	53	3	p	p	3	p	p	3
37	3	p	p	3	p	43	3	7	p	3	p	p	3	47	7	3	p	37	3	31
39	p	3	p	p	3	p	7	3	17	p	3	43	41	3	19	p	3	p	11	3
41	13	p	3	p	p	3	19	p	3	17	p	3	7	13	3	p	11	3	23	7
43	3	p	p	3	7	p	3	13	p	3	17	7	3	p	11	3	p	19	3	p
47	23	19	3	p	p	3	p	41	3	7	11	3	17	p	3	p	7	3	p	p
49	3	7	13	3	31	p	3	p	7	3	p	47	3	17	p	3	41	23	3	11
51	7	3	p	p	3	p	11	3	p	13	3	23	p	3	7	53	3	11	p	3
53	p	p	3	13	11	3	7	p	3	p	43	3	p	7	3	11	13	3	p	59
57	11	3	37	p	3	p	p	3	p	p	3	7	p	3	p	p	3	13	7	3
59	29	17	3	7	p	3	p	31	3	11	7	3	p	p	3	p	p	3	17	37
61	3	p	7	3	23	13	3	11	p	3	p	29	3	p	p	3	7	p	3	17
63	p	3	31	17	3	11	p	3	7	p	3	p	13	3	p	7	3	53	p	3
67	3	11	p	3	p	17	3	p	47	3	p	p	3	7	p	3	19	p	3	p
69	p	3	p	23	3	7	17	3	19	p	3	p	7	3	p	43	3	p	53	3
71	19	13	3	p	7	3	p	17	3	p	37	3	p	p	3	p	p	3	7	11
73	3	41	p	3	p	31	3	47	13	3	7	19	3	p	23	3	p	7	3	29
77	31	7	3	p	p	3	p	p	3	13	17	3	29	11	3	7	p	3	p	41
79	3	p	43	3	37	p	3	7	p	3	p	11	3	31	7	3	13	p	3	23
81	p	3	p	p	3	29	7	3	43	11	3	p	17	3	59	p	3	19	p	3
83	p	37	3	p	13	3	p	11	3	19	p	3	7	17	3	p	29	3	11	7
87	p	3	p	7	3	13	p	3	p	29	3	p	19	3	11	17	3	7	13	3
89	p	11	3	p	19	3	p	p	3	7	p	3	11	p	3	37	7	3	p	p
91	3	7	29	3	47	p	3	p	7	3	11	p	3	p	p	3	p	17	3	13
93	7	3	p	p	3	p	p	3	11	41	3	31	37	3	7	p	3	p	17	3
97	3	13	p	3	11	7	3	p	p	3	19	23	3	43	13	3	p	p	3	7
99	p	3	11	p	3	23	p	3	13	p	3	7	p	3	p	59	3	29	7	3

	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
01	p	3	p	11	3	7	43	3	p	13	3	p	7	3	11	p	3	p	p	3
03	p	11	3	13	7	3	p	p	3	p	p	3	11	p	3	p	13	3	7	p
07	p	3	7	59	3	p	17	3	11	7	3	p	41	3	p	p	3	13	p	3
09	19	7	3	31	p	3	11	17	3	p	p	3	p	p	3	7	71	3	37	19
11	3	p	p	3	11	13	3	7	17	3	p	19	3	47	7	3	31	p	3	23
13	p	3	11	19	3	p	7	3	p	17	3	p	13	3	p	37	3	29	p	3
17	3	23	p	3	7	p	3	53	p	3	29	7	3	13	p	3	41	p	3	61
19	p	3	p	7	3	p	31	3	61	p	3	p	17	3	p	p	3	7	11	3
21	p	13	3	29	p	3	p	p	3	7	p	3	23	17	3	p	7	3	p	31
23	3	7	41	3	p	p	3	p	7	3	p	47	3	p	11	3	p	59	3	p
27	p	p	3	p	19	3	7	29	3	13	11	3	p	7	3	p	17	3	p	p
29	3	p	p	3	43	7	3	p	11	3	47	23	3	73	61	3	13	17	3	7
31	29	3	p	61	3	23	11	3	p	p	3	7	p	3	p	p	3	11	7	3
33	37	p	3	7	11	3	41	p	3	p	7	3	p	p	3	11	43	3	19	17
37	11	3	19	p	3	13	p	3	7	p	3	11	p	3	p	7	3	p	13	3
39	7	p	3	p	23	3	p	7	3	11	p	3	13	19	3	29	p	3	p	p
41	3	41	p	3	p	19	3	11	47	3	71	53	3	7	p	3	p	p	3	13
43	13	3	p	43	3	7	p	3	29	p	3	37	7	3	p	23	3	p	p	3
47	3	11	31	3	p	p	3	47	37	3	7	p	3	p	13	3	p	7	3	19
49	p	3	7	p	3	p	p	3	13	7	3	19	29	3	p	31	3	p	p	3
51	p	7	3	19	p	3	p	p	3	p	p	3	59	p	3	7	p	3	p	11
53	3	p	p	3	61	29	3	7	23	3	31	p	3	53	7	3	p	11	3	p
57	p	p	3	p	p	3	p	67	3	p	13	3	7	11	3	p	p	3	p	7
59	3	p	p	3	7	47	3	p	43	3	p	7	3	23	53	3	p	13	3	59
61	31	3	p	7	3	p	59	3	p	11	3	13	p	3	43	67	3	7	p	3
63	17	23	3	p	p	3	p	11	3	7	61	3	19	31	3	p	7	3	11	67
67	7	3	17	11	3	p	13	3	31	p	3	p	23	3	7	19	3	73	p	3
69	13	11	3	17	41	3	7	19	3	p	37	3	11	7	3	p	p	3	p	47
71	3	43	p	3	17	7	3	13	p	3	11	p	3	41	p	3	53	29	3	7
73	p	3	p	p	3	17	p	3	11	p	3	7	p	3	13	p	3	23	7	3
77	3	p	7	3	11	23	3	17	p	3	p	31	3	19	p	3	7	53	3	43
79	p	3	11	29	3	19	p	3	7	13	3	p	p	3	p	p	3	p	p	3
81	7	37	3	13	p	3	31	7	3	17	p	3	p	p	3	p	13	3	p	p
83	3	47	p	3	p	p	3	p	19	3	13	71	3	7	p	3	p	p	3	31
87	61	53	3	41	7	3	43	p	3	p	p	3	17	p	3	37	11	3	7	p
89	3	59	p	3	67	13	3	p	p	3	7	p	3	17	11	3	p	7	3	53
91	p	3	7	p	3	p	p	3	67	7	3	29	11	3	17	p	3	p	43	3
93	p	7	3	23	p	3	13	p	3	p	11	3	67	p	3	7	p	3	71	13
97	17	3	p	p	3	p	7	3	59	19	3	p	p	3	23	29	3	11	p	3
99	p	13	3	53	11	3	37	p	3	p	p	3	7	p	3	11	41	3	17	7

	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
01	17	p	3	p	37	3	7	p	3	67	p	3	19	7	3	13	11	3	29	p
03	3	17	p	3	19	7	3	p	p	3	47	p	3	67	11	3	p	p	3	7
07	p	31	3	7	43	3	p	19	3	p	7	3	p	p	3	p	p	3	37	p
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87	p	3	7	p	3	29	19	3	p	7	3	p	p	3	17	11	3	p	p	3
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87	11	3	83	139	3	p	p	3	p	p	3	7	p	3	p	p	3	13	7	3
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93	67	3	43	29	3	11	p	3	7	p	3	19	13	3	p	7	3	47	p	3
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63	3	11	13	3	p	31	3	7	p	3	p	17	3	103	7	3	47	107	3	p
67	23	149	3	13	p	3	p	p	3	p	p	3	7	17	3	19	13	3	p	7
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03	79	3	p	7	3	191	13	3	11	83	3	23	17	3	p	p	3	7	163	3
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67	3	29	p	3	53	41	3	89	p	3	11	31	3	7	19	3	p	p	3	43
69	127	3	p	13	3	7	19	3	11	193	3	17	7	3	41	p	3	37	p	3
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03	179	p	3	19	7	3	29	p	3	17	11	3	13	p	3	67	181	3	7	p
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61	p	3	p	7	3	101	29	3	p	151	3	p	167	3	31	199	3	7	11	3
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91	p	3	7	23	3	p	p	3	13	7	3	41	19	3	p	p	3	p	83	3
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03	3	11	19	3	97	7	3	113	37	3	p	p	3	47	127	3	p	23	3	7
07	61	73	3	7	p	3	13	53	3	p	7	3	p	p	3	31	113	3	p	11
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87	p	3	p	p	3	p	7	3	151	67	3	17	p	3	p	79	3	p	11	3
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01	149	3	p	p	3	p	23	3	7	p	3	p	p	3	p	7	3	83	11	3
03	7	p	3	193	13	3	41	7	3	p	p	3	83	151	3	p	11	3	173	19
07	131	3	17	19	3	7	31	3	p	191	3	23	7	3	p	p	3	43	13	3
09	p	107	3	17	7	3	p	p	3	157	11	3	13	p	3	73	p	3	7	31
11	3	31	109	3	17	p	3	p	11	3	7	173	3	89	p	3	p	7	3	11
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27	p	p	3	11	103	3	p	p	3	7	13	3	17	p	3	p	7	3	19	p
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37	17	3	p	199	3	107	13	3	p	p	3	7	139	3	p	11	3	17	7	3
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43	71	3	89	17	3	p	61	3	7	11	3	19	37	3	13	7	3	223	23	3
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53	3	p	p	3	p	p	3	71	17	3	7	23	3	p	p	3	p	7	3	163
57	p	7	3	41	p	3	11	p	3	p	17	3	19	229	3	7	p	3	p	79
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71	3	7	167	3	137	p	3	113	7	3	73	p	3	19	11	3	191	17	3	31
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81	p	p	3	7	11	3	139	47	3	p	7	3	p	p	3	11	p	3	p	23
83	3	p	7	3	31	p	3	p	p	3	109	13	3	11	79	3	7	p	3	37
87	7	23	3	p	73	3	19	7	3	11	p	3	13	197	3	41	37	3	p	p
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93	113	19	3	11	7	3	23	13	3	197	p	3	137	107	3	p	p	3	7	p
97	59	3	7	151	3	149	p	3	13	7	3	p	223	3	61	p	3	23	p	3
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01	p	p	3	13	p	3	p	19	3	7	p	3	p	17	3	p	7	3	41	p
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27	3	113	211	3	37	11	3	p	109	3	7	p	3	61	43	3	11	7	3	p
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63	3	p	11	3	107	p	3	23	83	3	17	p	3	7	37	3	p	p	3	191
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37	p	7	3	p	13	3	p	p	3	p	67	3	11	83	3	7	p	3	p	241
39	3	p	59	3	19	p	3	7	83	3	11	13	3	p	7	3	53	107	3	23
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03	p	3	17	p	3	7	p	3	13	p	3	p	7	3	19	11	3	p	p	3
07	3	173	p	3	17	p	3	73	181	3	7	11	3	29	163	3	p	7	3	p
09	59	3	7	13	3	17	137	3	107	7	3	223	31	3	p	41	3	p	p	3
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07	p	3	11	107	3	251	23	3	229	47	3	7	197	3	p	13	3	p	7	3
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19	p	3	23	19	3	97	p	3	7	p	3	p	229	3	p	7	3	p	11	3
21	7	p	3	p	13	3	p	7	3	p	29	3	67	73	3	37	11	3	p	23
23	3	p	p	3	p	109	3	197	p	3	p	13	3	7	11	3	67	17	3	71
27	239	23	3	p	7	3	p	107	3	19	11	3	13	p	3	p	41	3	7	17
29	3	19	p	3	p	p	3	p	11	3	7	p	3	p	p	3	83	7	3	11
31	13	3	7	53	3	251	11	3	193	7	3	83	19	3	61	233	3	11	109	3
33	59	7	3	61	11	3	23	13	3	89	251	3	p	p	3	7	p	3	29	p
37	11	3	p	37	3	p	7	3	13	p	3	11	p	3	p	p	3	23	p	3
39	p	p	3	31	p	3	p	127	3	11	p	3	7	p	3	13	71	3	19	7
41	3	p	p	3	7	23	3	11	p	3	19	7	3	p	199	3	31	p	3	p
43	89	3	19	7	3	11	41	3	p	61	3	p	191	3	p	29	3	7	p	3
47	3	7	199	3	13	19	3	263	7	3	23	p	3	p	37	3	p	13	3	p
49	7	3	p	103	3	p	31	3	p	p	3	13	p	3	7	p	3	157	p	3
51	p	29	3	p	p	3	7	139	3	p	227	3	43	7	3	p	137	3	13	11
53	3	31	163	3	47	7	3	p	p	3	41	p	3	p	p	3	79	11	3	7
57	13	p	3	7	p	3	p	173	3	p	7	3	p	11	3	163	131	3	181	47
59	3	17	7	3	p	37	3	13	59	3	p	11	3	p	19	3	7	73	3	227
61	p	3	17	71	3	41	19	3	7	11	3	p	p	3	13	7	3	p	p	3
63	7	p	3	17	31	3	p	7	3	29	179	3	p	p	3	p	p	3	11	p
67	p	3	29	11	3	7	p	3	p	13	3	p	7	3	11	59	3	43	p	3
69	41	11	3	13	7	3	17	p	3	p	p	3	11	23	3	p	13	3	7	79
71	3	47	p	3	19	p	3	17	131	3	7	p	3	149	p	3	p	7	3	p
73	79	3	7	p	3	p	29	3	11	7	3	103	263	3	p	19	3	13	41	3
77	3	p	31	3	11	13	3	7	p	3	17	109	3	137	7	3	229	p	3	167
79	p	3	11	p	3	163	7	3	p	p	3	17	13	3	p	31	3	179	p	3
81	11	p	3	p	p	3	13	37	3	167	p	3	7	41	3	47	43	3	p	7
83	3	p	67	3	7	p	3	p	73	3	31	7	3	13	p	3	97	23	3	167
87	109	13	3	59	p	3	p	71	3	7	67	3	p	p	3	17	7	3	p	p
89	3	7	p	3	p	p	3	29	7	3	p	257	3	p	11	3	17	p	3	193
91	7	3	13	43	3	73	223	3	p	p	3	p	11	3	7	13	3	17	29	3
93	29	17	3	p	157	3	7	p	3	13	11	3	p	7	3	p	p	3	17	p
97	191	3	p	17	3	227	11	3	31	p	3	7	83	3	19	p	3	11	7	3
99	p	p	3	7	11	3	19	83	3	p	7	3	37	p	3	11	p	3	p	p

	720	721	722	723	724	725	726	727	728	729	730	731	732	733	734	735	736	737	738	739
01	89	p	3	17	7	3	97	p	3	p	37	3	71	23	3	31	11	3	7	67
03	3	p	103	3	17	p	3	23	47	3	7	41	3	p	11	3	89	7	3	263
07	13	7	3	p	61	3	17	p	3	p	11	3	19	13	3	7	p	3	23	p
09	3	p	163	3	19	31	3	7	11	3	p	29	3	p	7	3	p	p	3	11
11	107	3	p	167	3	59	7	3	17	p	3	113	179	3	13	19	3	11	31	3
13	23	37	3	p	11	3	p	19	3	17	p	3	7	167	3	11	p	3	223	p
17	11	3	257	7	3	127	p	3	p	13	3	11	211	3	p	p	3	7	97	3
19	p	41	3	13	139	3	101	p	3	7	p	3	17	157	3	37	7	3	p	193
21	3	7	p	3	p	47	3	11	7	3	13	p	3	17	p	3	83	p	3	29
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39	3	p	29	3	107	17	3	p	13	3	p	11	3	7	23	3	211	19	3	p
41	61	3	13	p	3	7	17	3	23	11	3	p	7	3	271	13	3	37	41	3
43	p	19	3	73	7	3	p	11	3	13	p	3	p	71	3	251	p	3	7	p
47	p	3	7	11	3	p	p	3	97	7	3	193	89	3	11	p	3	29	p	3
49	109	7	3	71	13	3	p	23	3	p	17	3	11	41	3	7	47	3	p	73
51	3	23	p	3	53	p	3	7	263	3	7	13	3	p	7	3	p	p	3	p
53	p	3	p	p	3	13	7	3	11	p	3	191	17	3	p	p	3	131	13	3
57	3	59	19	7	7	73	3	31	41	3	43	7	3	109	17	3	73	p	3	13
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61	11	p	3	269	p	3	p	13	3	7	p	3	61	p	3	p	7	3	233	p
63	3	7	127	3	233	149	3	p	7	3	p	23	3	p	13	3	19	17	3	37
67	19	p	3	p	p	3	7	p	3	131	31	3	41	7	3	13	11	3	p	17
69	3	p	p	3	p	7	3	53	p	3	89	19	3	p	11	3	23	71	3	7
71	97	3	p	13	3	31	p	3	p	43	3	7	11	3	p	p	3	p	7	3
73	p	p	3	7	23	3	p	61	3	p	7	3	47	239	3	29	p	3	31	p
77	p	3	p	157	3	p	11	3	7	p	3	13	p	3	p	7	3	11	p	3
79	7	89	3	p	11	3	p	7	3	19	p	3	127	p	3	11	p	3	13	29
81	3	19	11	p	p	181	3	73	31	3	107	p	3	7	179	3	p	89	3	167
83	11	3	41	3	3	7	13	3	p	59	3	11	7	3	p	p	3	p	p	3
87	3	37	p	3	173	29	3	11	23	3	7	163	3	p	43	3	31	7	3	241
89	p	3	7	191	3	11	p	3	p	7	3	p	83	3	13	p	3	113	37	3
91	p	7	3	11	71	3	157	83	3	47	p	3	p	79	3	7	59	3	19	23
93	3	11	13	3	p	229	3	7	p	3	19	53	3	23	p	3	p	109	3	61
97	17	23	3	13	p	3	139	p	3	p	67	3	7	19	3	p	13	3	p	7
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	740	741	742	743	744	745	746	747	748	749	750	751	752	753	754	755	756	757	758	759
01	3	p	p	3	47	7	3	11	131	3	179	13	3	257	p	3	19	17	3	7
03	43	3	p	67	3	11	61	3	19	p	3	7	157	3	p	p	3	p	7	3
07	3	11	7	3	37	p	3	p	239	3	107	19	3	p	p	3	7	p	3	13
09	13	3	p	19	3	p	p	3	7	173	3	p	p	3	73	7	3	p	41	3
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13	3	13	47	3	p	269	3	p	79	3	p	31	3	7	13	3	83	11	3	p
17	p	137	3	p	7	3	29	p	3	19	p	3	p	11	3	13	p	3	7	89
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21	p	3	7	13	3	p	71	3	p	7	3	43	19	3	199	p	3	p	p	3
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27	p	3	199	11	3	p	7	3	p	31	3	13	p	3	11	p	3	41	191	3
29	181	11	3	239	263	3	37	p	3	p	p	3	7	p	3	47	p	3	13	7
31	3	p	p	3	7	p	3	p	p	3	11	7	3	71	p	3	53	p	3	p
33	101	3	19	7	3	73	13	3	11	p	3	p	23	3	241	p	3	7	p	3
37	3	7	61	3	11	19	3	13	7	3	p	227	3	p	p	3	43	53	3	p
39	7	3	11	79	3	131	101	3	67	137	3	29	p	3	7	p	3	23	181	3
41	11	151	3	17	p	3	7	31	3	p	p	3	61	7	3	p	p	3	149	p
43	3	p	13	3	17	7	3	41	p	3	101	163	3	59	37	3	67	p	3	7
47	p	53	3	7	109	3	17	p	3	149	7	3	47	p	3	31	11	3	73	173
49	3	p	7	3	p	127	3	17	29	3	13	p	3	151	11	3	7	211	3	53
51	p	3	41	149	3	p	19	3	7	241	3	223	11	3	197	7	3	13	101	3
53	7	29	3	p	p	3	p	7	3	17	11	3	p	p	3	p	p	3	p	151
57	103	3	p	p	3	7	11	3	p	23	3	17	7	3	61	p	3	11	31	3
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71	p	17	3	11	p	3	89	p	3	13	41	3	7	23	3	p	31	3	17	7
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79	3	7	p	3	71	17	3	p	7	3	p	13	3	43	p	3	p	11	3	p
81	7	3	59	p	3	13	17	3	103	97	3	p	83	3	7	11	3	p	13	3
83	23	31	3	p	211	3	7	17	3	167	p	3	13	7	3	p	p	3	p	p
87	13	3	p	73	3	p	p	3	p	11	3	7	79	3	19	131	3	p	7	3
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93	p	3	p	11	3	97	113	3	7	19	3	p	17	3	11	7	3	p	29	3
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	760	761	762	763	764	765	766	767	768	769	770	771	772	773	774	775	776	777	778	779
01	p	3	181	41	3	113	7	3	p	11	3	p	p	3	17	19	3	13	p	3
03	p	p	3	p	p	3	p	11	3	53	p	3	7	23	3	17	71	3	11	7
07	17	3	p	7	3	p	p	3	89	p	3	83	13	3	11	179	3	7	29	3
09	29	11	3	137	100	3	13	79	3	7	53	3	11	97	3	p	7	3	17	13
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31	p	3	p	37	3	7	p	3	p	19	3	137	7	3	p	31	3	p	13	3
33	139	19	3	p	7	3	197	p	3	107	11	3	13	17	3	23	29	3	7	p
37	13	3	7	23	3	p	11	3	p	7	3	p	p	3	211	17	3	11	277	3
39	p	7	3	97	11	3	173	13	3	47	41	3	p	p	3	7	17	3	p	59
41	3	13	11	3	p	p	3	7	43	3	p	p	3	11	7	3	p	17	3	41
43	11	3	p	p	3	p	7	3	13	p	3	11	p	3	43	p	3	p	17	3
47	3	p	19	3	7	41	3	11	p	3	p	7	3	p	p	3	p	p	3	23
49	113	3	p	7	3	11	p	3	31	p	3	179	p	3	41	p	3	7	p	3
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53	3	7	p	3	13	37	3	p	7	3	29	p	3	103	73	3	19	13	3	137
57	19	p	3	29	101	3	7	p	3	41	251	3	23	7	3	p	79	3	13	11
59	3	p	p	3	157	7	3	59	151	3	263	19	3	p	29	3	p	11	3	7
61	23	3	p	19	3	p	13	3	101	p	3	7	p	3	71	11	3	p	7	3
63	13	p	3	7	p	3	31	29	3	p	7	3	p	11	3	p	37	3	p	53
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71	3	19	13	3	p	11	3	p	p	3	37	p	3	7	p	3	11	83	3	103
73	127	3	89	11	3	7	p	3	p	13	3	229	7	3	11	p	3	p	43	3
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79	p	3	7	p	3	p	p	3	11	7	3	113	p	3	p	23	3	13	47	3
81	p	7	3	17	p	3	11	p	3	23	p	3	109	223	3	7	p	3	19	29
83	3	29	p	3	11	13	3	7	p	3	19	79	3	p	7	3	131	p	3	p
87	11	47	3	p	p	3	13	31	3	167	157	3	7	19	3	p	p	3	71	7
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91	p	3	23	7	3	191	53	3	17	p	3	p	p	3	p	p	3	7	11	3
93	47	13	3	79	p	3	271	41	3	7	p	3	37	193	3	31	7	3	p	23
97	7	3	13	241	3	p	p	3	131	37	3	17	11	3	7	13	3	p	61	3
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	780	781	782	783	784	785	786	787	788	789	790	791	792	793	794	795	796	797	798	799
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37	73	p	3	7	p	3	13	p	3	193	7	3	17	p	3	p	97	3	29	11
39	3	p	7	3	p	p	3	71	p	3	p	p	3	13	19	3	7	11	3	p
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43	7	13	3	157	47	3	p	7	3	89	p	3	109	11	3	17	73	3	p	p
47	17	3	13	p	3	7	31	3	37	11	3	p	7	3	53	13	3	17	p	3
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61	251	47	3	23	31	3	11	17	3	281	173	3	7	61	3	p	37	3	p	7
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67	11	p	3	p	p	3	97	13	3	7	17	3	31	p	3	251	7	3	p	p
69	3	7	23	3	131	p	3	227	7	3	37	17	3	139	13	3	p	p	3	211
71	7	3	29	109	3	p	151	3	13	157	3	41	17	3	7	47	3	241	11	3
73	101	p	3	181	97	3	7	37	3	151	107	3	p	7	3	13	11	3	p	p
77	163	3	p	13	3	p	29	3	p	p	3	7	11	3	19	17	3	p	7	3
79	p	p	3	7	p	3	19	p	3	p	7	3	p	p	3	p	17	3	23	p
81	3	37	7	3	13	179	3	p	11	3	31	p	3	163	p	3	7	13	3	11
83	113	3	p	103	3	p	11	3	7	19	3	13	p	3	61	7	3	11	17	3
87	3	41	11	3	p	89	3	p	p	3	p	p	3	7	101	3	p	23	3	p
89	11	3	79	43	3	7	13	3	p	p	3	11	7	3	29	p	3	73	p	3
91	13	p	3	277	7	3	p	p	3	11	139	3	37	13	3	19	p	3	7	41
93	3	p	59	3	53	p	3	11	p	3	7	p	3	p	p	3	p	7	3	167
97	29	7	3	11	p	3	p	p	3	197	19	3	179	p	3	7	p	3	109	p
99	3	p	13	3	23	53	3	7	257	3	83	29	3	p	7	3	p	199	3	p

	800	801	802	803	804	805	806	807	808	809	810	811	812	813	814	815	816	817	818	819
01	3	7	11	3	37	79	3	p	7	3	p	p	3	11	p	3	13	p	3	p
03	7	3	139	131	3	19	p	3	p	17	3	11	p	3	7	149	3	p	179	3
07	3	p	p	3	p	7	3	11	19	3	59	13	3	p	127	3	79	p	3	7
09	19	3	p	p	3	11	149	3	p	p	3	7	17	3	p	p	3	101	7	3
11	29	p	3	7	191	3	p	43	3	p	7	3	13	17	3	37	p	3	23	101
13	3	11	7	3	97	p	3	p	211	3	p	29	3	31	17	3	7	41	3	13
17	7	113	3	p	29	3	19	7	3	p	p	3	241	233	3	p	17	3	p	11
19	3	13	97	3	137	73	3	53	p	3	p	p	3	7	13	3	p	11	3	p
21	p	3	p	31	3	7	p	3	13	19	3	23	7	3	p	11	3	71	17	3
23	43	19	3	47	7	3	37	89	3	p	p	3	p	11	3	13	31	3	7	17
27	79	3	7	13	3	p	p	3	131	7	3	31	43	3	107	p	3	p	47	3
29	191	7	3	p	p	3	p	11	3	p	13	3	29	167	3	7	p	3	11	p
31	3	227	p	3	13	11	3	7	p	3	p	p	3	p	7	3	11	13	3	p
33	163	3	p	67	3	29	7	3	p	p	3	13	p	3	11	p	3	37	19	3
37	3	127	19	3	7	p	3	p	229	3	11	7	3	163	31	3	p	p	3	p
39	p	3	p	7	3	43	13	3	11	29	3	41	p	3	p	67	3	7	p	3
41	13	p	3	p	257	3	11	263	3	7	p	3	137	13	3	73	7	3	223	67
43	3	7	29	3	11	239	3	13	7	3	p	53	3	p	23	3	19	43	3	p
47	11	p	3	p	p	3	7	p	3	61	p	3	113	7	3	p	p	3	p	19
49	3	p	13	3	p	7	3	p	p	3	p	19	3	p	79	3	p	p	3	7
51	p	3	p	19	3	109	p	3	233	13	3	7	31	3	47	p	3	29	7	3
53	17	p	3	7	43	3	59	23	3	p	7	3	193	p	3	p	11	3	p	p
57	223	3	17	107	3	p	p	3	7	73	3	p	11	3	p	7	3	13	23	3
59	7	71	3	17	61	3	79	7	3	19	11	3	23	p	3	p	37	3	109	41
61	3	19	83	3	17	13	3	p	11	3	103	277	3	7	29	3	127	p	3	11
63	23	3	p	p	3	7	p	3	p	p	3	p	7	3	p	p	3	11	71	3
67	3	p	11	3	67	p	3	17	193	3	7	23	3	11	41	3	p	7	3	p
69	11	3	7	p	3	23	p	3	17	7	3	11	181	3	257	p	3	p	p	3
71	p	7	3	179	p	3	p	37	3	11	p	3	67	p	3	7	p	3	19	p
73	3	p	p	3	p	197	3	7	13	3	17	p	3	p	7	3	23	p	3	p
77	p	p	3	11	23	3	p	p	3	13	p	3	7	19	3	29	p	3	41	7
79	3	11	p	3	7	19	3	p	31	3	89	7	3	17	59	3	13	53	3	73
81	73	3	43	7	3	61	p	3	29	47	3	p	p	3	17	23	3	7	37	3
83	53	181	3	31	13	3	p	p	3	7	p	3	p	97	3	17	7	3	p	11
87	7	3	p	p	3	13	p	3	47	109	3	19	29	3	7	11	3	17	13	3
89	283	17	3	19	p	3	7	p	3	p	131	3	13	7	3	83	p	3	17	163
91	3	p	17	3	p	7	3	173	23	3	83	11	3	199	19	3	151	89	3	7
93	13	3	23	17	3	83	19	3	41	11	3	7	p	3	227	139	3	263	7	3
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99	173	3	59	11	3	p	17	3	7	107	3	p	p	3	11	7	3	p	p	3

	820	821	822	823	824	825	826	827	828	829	830	831	832	833	834	835	836	837	838	839
01	43	3	7	p	3	17	p	3	31	7	3	p	19	3	p	11	3	p	47	3
03	p	7	3	13	19	3	17	191	3	p	p	3	p	11	3	7	13	3	181	p
07	p	3	p	p	3	p	7	3	17	11	3	41	p	3	p	113	3	13	43	3
09	p	47	3	53	23	3	p	11	3	17	p	3	7	227	3	37	p	3	11	7
11	3	157	229	3	7	11	3	107	p	3	17	7	3	p	239	3	11	97	3	p
13	p	3	19	7	3	109	p	3	p	p	3	17	13	3	11	23	3	7	p	3
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31	p	3	p	17	3	p	19	3	7	127	3	59	p	3	p	7	3	31	11	3
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37	p	3	p	137	3	7	17	3	p	197	3	p	7	3	p	p	3	p	13	3
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41	3	p	p	3	19	59	3	97	11	3	7	71	3	p	181	3	p	7	3	11
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47	3	13	11	3	29	23	3	7	p	3	p	17	3	11	7	3	233	83	3	127
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57	31	29	3	11	p	3	p	p	3	7	13	3	p	p	3	p	7	3	p	59
59	3	7	43	3	13	p	3	p	7	3	p	137	3	31	p	3	269	13	3	113
61	7	3	p	p	3	p	131	3	41	23	3	13	139	3	7	p	3	p	17	3
63	137	p	3	23	p	3	7	p	3	p	p	3	53	7	3	p	p	3	13	11
67	p	3	p	31	3	p	13	3	173	163	3	7	p	3	19	11	3	211	7	3
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71	3	p	7	3	p	p	3	13	79	3	p	11	3	263	p	3	7	19	3	131
73	p	3	29	p	3	71	47	3	7	11	3	31	p	3	13	7	3	p	p	3
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79	211	3	p	11	3	7	29	3	67	13	3	223	7	3	11	p	3	199	37	3
81	79	11	3	13	7	3	89	p	3	p	251	3	11	199	3	19	13	3	7	137
83	3	p	107	3	p	269	3	19	p	3	7	193	3	p	31	3	67	7	3	p
87	23	7	3	p	p	3	11	p	3	31	19	3	37	61	3	7	53	3	149	p
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91	103	3	11	47	3	p	7	3	p	37	3	23	13	3	29	p	3	p	p	3
93	11	p	3	p	p	3	13	p	3	149	p	3	7	89	3	179	127	3	43	7
97	53	3	17	7	3	151	41	3	19	p	3	271	31	3	p	p	3	7	11	3
99	19	13	3	17	p	3	p	p	3	7	23	3	p	p	3	41	7	3	53	19

	8400	8410	8420	8430	8440	8450	8460	8470	8480	8490	8500	8510	8520	8530	8540	8550	8560	8570	8580	8590
01	167	37	3	7	p	3	11	p	3	59	7	3	p	197	3	13	p	3	239	17
03	3	31	7	3	11	p	3	71	137	3	167	p	3	p	41	3	7	p	3	p
07	7	151	3	p	p	3	19	7	3	197	13	3	139	23	3	37	p	3	53	271
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37	19	p	3	11	p	3	7	p	3	157	p	3	p	7	3	23	29	3	p	19
39	3	11	p	3	17	7	3	101	43	3	277	19	3	61	p	3	p	83	3	7
41	31	3	61	19	3	17	53	3	37	29	3	7	13	3	43	113	3	179	7	3
43	229	p	3	7	p	3	13	83	3	173	7	3	p	31	3	131	p	3	p	11
47	p	3	p	p	3	59	47	3	7	p	3	p	p	3	p	7	3	19	p	3
49	7	13	3	p	p	3	p	7	3	17	p	3	163	11	3	p	41	3	293	61
51	3	19	173	3	79	p	3	p	13	3	17	11	3	7	p	3	97	p	3	23
53	p	3	13	67	3	7	p	3	53	11	3	17	7	3	p	13	3	29	p	3
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63	3	p	p	3	p	103	3	7	113	3	11	13	3	p	7	3	17	139	3	31
67	p	17	3	239	p	3	11	29	3	p	257	3	7	19	3	41	p	3	17	7
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71	13	3	11	7	3	23	227	3	p	31	3	53	71	3	127	p	3	7	43	3
73	11	41	3	139	17	3	p	13	3	7	241	3	269	59	3	83	7	3	79	149
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79	83	p	3	19	23	3	7	17	3	p	149	3	107	7	3	13	11	3	157	127
81	3	p	271	3	p	7	3	149	17	3	p	103	3	p	11	3	47	p	3	7
83	47	3	89	13	3	41	19	3	29	17	3	7	11	3	73	23	3	109	7	3
87	3	29	7	3	13	251	3	p	11	3	p	17	3	103	p	3	7	13	3	11
89	p	3	31	p	3	p	11	3	7	37	3	13	17	3	53	7	3	11	p	3
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93	3	59	11	3	19	29	3	p	23	3	p	p	3	7	17	3	67	p	3	113
97	13	269	3	37	7	3	p	19	3	11	43	3	p	13	3	p	17	3	7	23
99	3	p	p	3	p	31	3	11	73	3	7	p	3	23	193	3	43	7	3	p

	860	861	862	863	864	865	866	867	868	869	870	871	872	873	874	875	876	877	878	879
01	3	29	p	3	7	p	3	277	11	3	19	7	3	67	71	3	17	p	3	11
03	17	3	13	7	3	23	11	3	61	43	3	p	29	3	p	13	3	7	p	3
07	3	7	11	3	71	19	3	31	7	3	167	p	3	11	p	3	13	229	3	17
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17	p	p	3	7	103	3	37	17	3	23	7	3	13	p	3	p	41	3	137	p
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21	13	3	151	37	3	31	19	3	7	17	3	p	p	3	p	7	3	p	53	3
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33	227	3	7	13	3	p	41	3	71	7	3	p	83	3	p	17	3	59	p	3
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39	97	3	p	11	3	p	7	3	37	p	3	13	23	3	11	p	3	p	17	3
41	139	11	3	p	p	3	23	127	3	227	p	3	7	167	3	p	p	3	13	7
43	3	p	p	3	7	37	3	p	p	3	11	7	3	19	p	3	p	p	3	p
47	13	277	3	79	137	3	11	223	3	7	61	3	43	13	3	p	7	3	107	31
49	3	7	p	3	11	23	3	13	7	3	p	p	3	113	157	3	p	47	3	37
51	7	3	11	p	3	41	73	3	p	p	3	p	p	3	7	29	3	p	59	3
53	11	101	3	p	p	3	7	p	3	89	263	3	p	7	3	p	23	3	p	281
57	47	3	p	p	3	101	193	3	p	13	3	7	p	3	19	p	3	127	7	3
59	41	29	3	7	31	3	19	101	3	p	7	3	71	p	3	p	11	3	103	p
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71	17	p	3	p	7	3	13	p	3	29	p	3	197	41	3	11	p	3	7	13
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77	p	7	3	17	p	3	p	107	3	11	19	3	p	23	3	7	43	3	p	p
79	3	p	19	3	17	p	3	7	13	3	31	p	3	59	7	3	p	61	3	97
81	59	3	13	p	3	11	7	3	283	p	3	p	p	3	p	13	3	41	p	3
83	p	p	3	11	197	3	17	p	3	13	p	3	7	p	3	p	p	3	23	7
87	31	3	p	7	3	p	23	3	17	37	3	p	191	3	89	p	3	7	p	3
89	19	79	3	p	13	3	p	59	3	7	73	3	41	31	3	p	7	3	179	11
91	3	7	p	3	p	131	3	229	7	3	17	13	3	281	p	3	p	11	3	p
93	7	3	p	19	3	13	p	3	31	p	3	17	p	3	7	11	3	p	13	3
97	3	p	p	3	67	7	3	29	113	3	251	11	3	17	59	3	p	p	3	7
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	880	881	882	883	884	885	886	887	888	889	890	891	892	893	894	895	896	897	898	899
01	p	3	193	p	3	7	41	3	p	19	3	p	7	3	13	p	3	271	89	3
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21	23	11	3	p	29	3	13	p	3	7	p	3	11	179	3	p	7	3	p	13
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27	19	13	3	p	p	3	7	83	3	17	127	3	p	7	3	p	p	3	43	19
29	3	p	83	3	11	7	3	p	13	3	17	19	3	p	37	3	47	53	3	7
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37	p	3	p	p	3	29	151	3	7	p	3	p	p	3	17	7	3	19	11	3
39	7	53	3	p	13	3	137	7	3	19	269	3	233	41	3	17	11	3	p	p
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43	17	3	79	23	3	7	p	3	p	29	3	97	7	3	p	151	3	17	13	3
47	3	181	17	3	241	p	3	p	11	3	7	239	3	47	23	3	157	7	3	11
49	13	3	7	17	3	73	11	3	23	7	3	59	31	3	p	149	3	11	p	3
51	191	7	3	53	11	3	p	13	3	p	p	3	149	199	3	7	37	3	19	293
53	3	13	11	3	197	17	3	7	p	3	19	p	3	11	7	3	p	p	3	23
57	173	199	3	149	53	3	p	17	3	11	p	3	7	19	3	13	p	3	59	7
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61	107	3	p	7	3	11	p	3	p	17	3	163	p	3	137	p	3	7	23	3
63	83	131	3	11	p	3	p	37	3	7	13	3	23	p	3	p	7	3	73	p
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73	29	3	41	67	3	23	13	3	p	193	3	7	p	3	131	11	3	107	7	3
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93	p	7	3	37	p	3	11	p	3	p	41	3	p	p	3	7	257	3	241	31
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01	p	11	3	73	p	3	7	13	3	p	17	3	11	7	3	37	139	3	p	29
03	3	13	p	3	p	7	3	p	p	3	11	17	3	p	13	3	47	p	3	7
07	p	p	3	7	p	3	11	61	3	p	7	3	223	17	3	13	101	p	p	73
09	3	251	7	3	11	29	3	p	71	3	p	31	3	p	17	3	7	293	3	p
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37	179	23	3	13	p	3	233	31	3	7	59	3	p	149	3	239	7	3	p	89
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47	53	3	p	167	3	p	p	3	p	p	3	7	13	3	19	43	3	23	7	3
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53	p	3	17	p	3	83	269	3	7	19	3	p	p	3	p	7	3	p	31	3
57	3	89	43	3	17	137	3	47	13	3	23	11	3	7	p	3	151	p	3	p
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67	p	7	3	23	13	3	71	139	3	17	19	3	11	p	3	7	31	3	p	p
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47	83	11	3	p	193	3	p	163	3	41	p	3	7	17	3	139	37	3	13	7
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01	23	3	p	181	3	11	13	3	7	43	3	p	31	3	p	7	3	p	p	3
03	7	139	3	11	67	3	p	7	3	p	p	3	p	13	3	43	p	3	p	29
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51	163	p	3	p	7	3	p	41	3	p	11	3	13	97	3	19	p	3	7	229
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93	23	11	3	13	7	3	p	p	3	p	p	3	11	p	3	109	13	3	7	59
97	73	3	7	p	3	p	281	3	11	7	3	23	233	3	29	p	3	p	17	3
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41	11	3	157	7	3	29	241	3	113	13	3	11	p	3	p	103	3	7	p	3
43	p	79	3	13	p	3	p	89	3	7	53	3	47	311	3	23	7	3	p	p
47	7	3	109	23	3	11	127	3	p	29	3	19	31	3	7	p	3	13	p	3
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67	17	p	3	29	7	3	p	11	3	13	113	3	23	p	3	43	101	3	7	p
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71	23	3	7	11	3	269	p	3	73	7	3	p	211	3	11	p	3	p	p	3
73	191	7	3	17	13	3	277	29	3	p	p	3	11	p	3	7	p	3	97	p
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83	13	3	11	7	3	59	109	3	17	293	3	157	p	3	71	p	3	7	p	3
87	3	7	73	3	p	p	3	p	7	3	17	p	3	p	13	3	p	p	3	p
89	7	3	p	113	3	p	31	3	13	p	3	17	271	3	7	23	3	p	11	3
91	307	43	3	41	47	3	7	151	3	23	79	3	17	7	3	13	11	3	53	29
93	3	29	p	3	p	7	3	43	p	3	151	83	3	17	11	3	211	19	3	7
97	p	19	3	7	p	3	p	p	3	p	7	3	149	p	3	17	151	3	223	43
99	3	p	7	3	13	29	3	p	11	3	89	37	3	173	p	3	7	13	3	11

	980	981	982	983	984	985	986	987	988	989	990	991	992	993	994	995	996	997	998	999
01	3	p	283	3	19	13	3	89	p	3	7	113	3	199	p	3	103	7	3	p
03	23	3	7	197	3	137	151	3	29	7	3	p	13	3	107	19	3	179	11	3
07	3	17	p	3	p	p	3	7	p	3	181	23	3	13	7	3	p	p	3	p
09	p	3	17	37	3	23	7	3	p	p	3	p	11	3	p	151	3	p	p	3
11	p	13	3	17	p	3	31	p	3	p	11	3	7	47	3	191	p	3	151	7
13	3	41	p	3	7	29	3	p	11	3	p	7	3	19	89	3	23	p	3	11
17	p	59	3	p	11	3	17	p	3	7	p	3	47	p	3	11	7	3	p	41
19	3	7	11	3	p	p	3	17	7	3	83	p	3	11	37	3	13	p	3	163
21	7	3	p	p	3	83	p	3	17	31	3	11	313	3	7	23	3	p	173	3
23	83	p	3	p	13	3	7	269	3	11	p	3	p	7	3	p	p	3	p	p
27	61	3	p	p	3	11	p	3	37	p	3	7	67	3	19	p	3	31	7	3
29	167	p	3	7	p	3	19	p	3	p	7	3	13	71	3	p	67	3	p	p
31	3	11	7	3	257	37	3	p	23	3	167	p	3	17	p	3	7	19	3	13
33	13	3	23	107	3	p	53	3	7	19	3	p	p	3	17	7	3	p	p	3
37	3	13	193	3	173	211	3	p	p	3	97	p	3	7	13	3	17	11	3	37
39	17	3	31	29	3	7	p	3	13	p	3	p	7	3	p	11	3	17	p	3
41	p	17	3	43	7	3	p	293	3	163	p	3	p	11	3	13	37	3	7	139
43	3	p	17	3	p	p	3	19	97	3	7	11	3	41	17	3	p	7	3	17
47	p	7	3	p	17	3	23	11	3	p	13	3	61	p	3	7	251	3	11	89
49	3	61	19	3	13	11	3	7	p	3	37	p	3	p	7	3	11	13	3	127
51	71	3	p	11	3	39	7	3	41	53	3	13	p	3	11	p	3	23	31	3
53	31	11	3	59	p	3	47	17	3	p	p	3	7	73	3	113	227	3	13	7
57	p	3	p	7	3	67	13	3	11	17	3	229	p	3	271	29	3	7	61	3
59	13	103	3	41	p	3	11	61	3	7	17	3	p	13	3	p	7	3	p	19
61	3	7	97	3	11	p	3	13	7	3	23	17	3	67	79	3	p	p	3	p
63	7	3	11	19	3	p	p	3	109	p	3	53	17	3	7	p	3	67	37	3
67	3	89	13	3	p	7	3	283	p	3	157	131	3	p	17	3	p	p	3	7
69	281	3	p	p	3	241	p	3	p	13	3	7	53	3	p	17	3	19	7	3
71	101	127	3	7	59	3	79	43	3	19	7	3	37	p	3	p	11	3	p	p
73	3	19	7	3	p	p	3	p	p	3	13	p	3	43	11	3	7	17	3	257
77	7	31	3	p	19	3	101	7	3	29	11	3	p	p	3	p	263	3	p	17
79	3	p	23	3	p	13	3	p	11	3	p	41	3	7	31	3	p	113	3	11
81	p	3	29	131	3	7	11	3	61	p	3	p	7	3	53	p	3	11	p	3
83	43	47	3	37	7	3	13	173	3	31	p	3	101	23	3	11	83	3	7	13
87	11	3	7	p	3	311	29	3	p	7	3	11	43	3	p	53	3	p	59	3
89	47	7	3	p	149	3	p	223	3	11	p	3	p	19	3	7	p	3	23	p
91	3	149	227	3	p	19	3	7	13	3	197	p	3	p	7	3	131	73	3	p
93	233	3	13	61	3	11	7	3	p	p	3	281	31	3	37	13	3	p	191	3
97	3	11	p	3	7	p	3	31	p	3	41	7	3	p	p	3	13	23	3	19
99	263	3	p	7	3	43	229	3	p	p	3	19	109	3	29	137	3	7	283	3

Of Rational Numbers that expreſs the Sides of Right-angled Triangles.

A PROBLEM.

Article 1. To find as many right-angled triangles as we pleaſe, of which the three ſides ſhall be expreſſible in rational numbers.

SOLUTION.

Let the numbers that expreſs the lengths of the two ſides that contain the right angle, be denoted by the letters m and n. Then will the number that denotes the hypotenuſe of the triangle, or the line that ſubtends the right angle, be greater than either of the two numbers m and n, and its exceſs above either of the ſaid numbers will be a rational number: for, if it were not, the number itſelf which expreſſes the ſaid hypotenuſe would not be a rational number. Let the exceſs of this number, which expreſſes the hypotenuſe, above the number m, which expreſſes one of the ſides containing the right angle, be called e. Then will the number which expreſſes the hypotenuſe be $m + e$, and its ſquare will be $mm + 2me + ee$. But (by El. 1, 47,) the ſquare of the hypotenuſe of a right-angled triangle is equal to the ſum of the ſquares of the two ſides of it. Therefore

$mm + 2me + ee$ will be equal to $mm + nn$; and, confequently (fubtracting mm from both fides), $2me + ee$ will be equal to nn; and (fubtracting ee from both fides, which is evidently lefs than $2me + ee$, and confequently muft be lefs alfo than nn, or the right-hand fide of the equation $2me + ee = nn$,) $2me$ will be equal to $nn - ee$; and (dividing both fides of the equation by $2e$,) m will be equal to $\frac{nn - ee}{2e}$, and confequently $m + e$ will be equal to $\frac{nn - ee}{2e} + e$, or to $\frac{nn - ee}{2e} + \frac{2e \times e}{2e}$, or to $\frac{nn - ee}{2e} + \frac{2ee}{2e}$, or to $\frac{nn + ee}{2e}$. And confequently the three numbers m, n, and $m + e$, that will exprefs the three fides of a right-angled triangle, will be equal to $\frac{nn - ee}{2e}$, n, and $\frac{nn + ee}{2e}$, or $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$; or, if we take any number whatfoever, and call it e, and take any other number whatfoever that is greater than e, and call it n, the three numbers $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$, will be three rational numbers that will exprefs the three fides of a right-angled triangle. Q. E. I.

Examples of this Method of finding fuch Rational Numbers.

Art. 2. Thus, for example, if e is $= 1$, and n is $= 2$, (which are the fimpleft numbers we can chufe,) we fhall have $ee = 1$, and $nn = 4$, and confequently $\frac{nn - ee}{2e}$ ($= \frac{4 - 1}{2 \times 1} = \frac{4 - 1}{2}$) $= \frac{3}{2}$, and $\frac{2en}{2e}$ ($= \frac{2 \times 1 \times 2}{2 \times 1}$) $= \frac{4}{2}$, and $\frac{nn + ee}{2e}$ ($= \frac{4 + 1}{2 \times 1} = \frac{4 + 1}{2}$) $= \frac{5}{2}$; and confequently

$\frac{3}{2}$,

$\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$, will be three rational numbers that will express the lengths of the three sides of a right-angled triangle. And accordingly we shall find that the square of the last of these numbers, to wit, $\frac{5}{2}$, which expresses the hypotenuse of the triangle, is equal to the sum of the squares of the two former numbers, to wit, $\frac{3}{2}$ and $\frac{4}{2}$, which express the two sides that contain the right angle. For the square of $\frac{5}{2}$ is $\frac{25}{4}$, and the squares of $\frac{3}{2}$ and $\frac{4}{2}$ are $\frac{9}{4}$ and $\frac{16}{4}$; and $\frac{25}{4}$ is $= \frac{9}{4} + \frac{16}{4}$.

Secondly, let e be $= 2$, and n be $= 3$.

Then we shall have $ee = 4$, and $nn = 9$, and $2e = 4$; and consequently $\frac{nn - ee}{2e}$ $(= \frac{9-4}{4}) = \frac{5}{4}$, and $\frac{2en}{2e}$ $(= \frac{2 \times 2 \times 3}{2 \times 2}) = \frac{12}{4}$, and $\frac{nn + ee}{2e}$ $(= \frac{9+4}{4}) = \frac{13}{4}$. Therefore $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$, will be three rational numbers that will express the three sides of a right-angled triangle. And accordingly we shall find that the square of $\frac{13}{4}$ will be equal to the sum of the squares of $\frac{5}{4}$ and $\frac{12}{4}$. For the square of $\frac{13}{4}$ is $= \frac{169}{16}$, and the squares of $\frac{5}{4}$ and $\frac{12}{4}$ are $\frac{25}{16}$ and $\frac{144}{16}$; and $\frac{169}{16}$ is $= \frac{25}{16} + \frac{144}{16}$.

Thirdly, let e be $= 3$, and n be $= 5$.

Then we shall have $ee = 9$, and $nn = 25$, and $2e = 6$, and consequently $\frac{nn - ee}{2e}$ $(= \frac{25-9}{6}) = \frac{16}{6}$, and $\frac{2en}{2e}$ $(= \frac{2 \times 3 \times 5}{6}) = \frac{30}{6}$, and $\frac{nn + ee}{2e}$ $(= \frac{25+9}{6}) = \frac{34}{6}$.

 Therefore

Therefore $\frac{16}{6}$, $\frac{30}{6}$, and $\frac{34}{6}$, will be three rational numbers that will expreſs the three ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of $\frac{34}{6}$ will be equal to the ſum of the ſquares of $\frac{16}{6}$ and $\frac{30}{6}$. For the ſquare of $\frac{34}{6}$ is $\frac{1156}{36}$, and the ſquares of $\frac{16}{6}$ and $\frac{30}{6}$ are $\frac{256}{36}$ and $\frac{900}{36}$; and $\frac{1156}{36}$ is $= \frac{256}{36} + \frac{900}{36}$.

Theſe three numbers $\frac{16}{6}$, $\frac{30}{6}$, and $\frac{34}{6}$, might have been reduced to ſmaller numbers, by dividing both their numerators and denominators by 2. For they would then have been $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$. Therefore theſe three numbers $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$, will expreſs the three ſides of a right-angled triangle.

Fourthly, let e be $= 3$, and $n = 7$.

Then we ſhall have $ee = 9$, and $nn = 49$, and $2e = 6$, and conſequently $\frac{nn - ee}{2e}$ $(= \frac{49 - 9}{6}) = \frac{40}{6}$, and $\frac{2en}{2e}$ $(= \frac{2 \times 3 \times 7}{6}) = \frac{42}{6}$, and $\frac{nn + ee}{2e}$ $(= \frac{49 + 9}{6}) = \frac{58}{6}$. Therefore $\frac{40}{6}$, $\frac{42}{6}$, and $\frac{58}{6}$, or $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$, will be three rational numbers that will expreſs the ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of $\frac{29}{3}$ will be equal to the ſum of the ſquares of $\frac{20}{3}$ and $\frac{21}{3}$. For the ſquare of $\frac{29}{3}$ is $\frac{841}{9}$, and the ſquares of $\frac{20}{3}$ and $\frac{21}{3}$ are $\frac{400}{9}$ and $\frac{441}{9}$; and $\frac{841}{9}$ is $= \frac{400}{9} + \frac{441}{9}$.

Fifthly,

Fifthly, let e be $= 3$, and $n = 11$.

Then we ſhall have $ee = 9$, and $nn = 121$, and $2e = 6$, and conſequently $\frac{nn - ee}{2e}$ $(= \frac{121 - 9}{6}) = \frac{112}{6}$, and $\frac{2en}{2e}$ $(= \frac{6 \times 11}{6}) = \frac{66}{6}$, and $\frac{nn + ee}{2e}$ $(= \frac{121 + 9}{6}) = \frac{130}{6}$. Therefore $\frac{112}{6}$, $\frac{66}{6}$, and $\frac{130}{6}$, or $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$, will be three rational numbers that will expreſs the three ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of $\frac{65}{3}$ will be equal to the ſum of the ſquares of $\frac{56}{3}$ and $\frac{33}{3}$. For the ſquare of $\frac{65}{3}$ is $\frac{4225}{9}$, and the ſquares of $\frac{56}{3}$ and $\frac{33}{3}$ are $\frac{3136}{9}$ and $\frac{1089}{9}$; and $\frac{4225}{9}$ is $= \frac{3136}{9} + \frac{1089}{9}$.

Sixthly, let e be $= 5$, and $n = 13$.

Then we ſhall have $ee = 25$, and $nn = 169$, and $2e = 10$, and conſequently $\frac{nn - ee}{2e}$ $(= \frac{169 - 25}{10}) = \frac{144}{10}$, and $\frac{2en}{2e}$ $(= \frac{10 \times 13}{10}) = \frac{130}{10}$, and $\frac{nn + ee}{2e}$ $(= \frac{169 + 25}{10}) = \frac{194}{10}$. Therefore $\frac{144}{10}$, $\frac{130}{10}$, and $\frac{194}{10}$, or $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$, will be three rational numbers that will expreſs the three ſides of a right-angled triangle. And accordingly we ſhall find that the ſquare of $\frac{97}{5}$ will be equal to the ſum of the ſquares of $\frac{72}{5}$ and $\frac{65}{5}$. For the ſquare of $\frac{97}{5}$ is $\frac{9409}{25}$, and the ſquares of $\frac{72}{5}$ and $\frac{65}{5}$ are $\frac{5184}{25}$ and $\frac{4225}{25}$; and $\frac{9409}{25}$ is $= \frac{5184}{25} + \frac{4225}{25}$.

Thus we have obtained ſix different ſets of rational numbers, which expreſs the lengths of the ſides of as many different

ferent right-angled triangles; to wit, 1st, $\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$; and, 2dly, $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$; and, 3dly, $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$; and, 4thly, $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$; and, 5thly, $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$; and, 6thly, $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$. And, by changing either both the numbers denoted by e and n, or only one of those numbers, and computing the values of the three fractions $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$, we may obtain as many more such sets of numbers as we please.

Art. 3. All these numbers are fractions, because they are derived from the general fractional expressions $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$. But, if we multiply the three fractions of each of these six sets of fractions by their common denominator, the products will be whole numbers expressing the sides of greater right-angled triangles similar to the former triangles, of which the sides were expressed by the foregoing fractions. Thus, if we multiply the three fractions $\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$, by their common denominator 2, we shall have the whole numbers 3, 4, and 5, for the sides of a greater right-angled triangle similar to the former triangle, of which the sides were expressed by the fractions $\frac{3}{2}$, $\frac{4}{2}$, and $\frac{5}{2}$. And, if we multiply the three fractions $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$, by their common denominator 4, we shall have the whole numbers 5, 12, and 13, for the sides of a greater right-angled triangle similar to the former triangle, of which the sides were $\frac{5}{4}$, $\frac{12}{4}$, and $\frac{13}{4}$. And, if we multiply the three fractions

$\frac{8}{3}$,

$\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$, by their common denominator 3, we ſhall have the whole numbers 8, 15, and 17, for the ſides of a greater right-angled triangle ſimilar to the former triangle, of which the ſides were $\frac{8}{3}$, $\frac{15}{3}$, and $\frac{17}{3}$. And, in like manner, from the fractions $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$, we may derive the whole numbers 20, 21, and 29; and from the fractions $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$, we may derive the whole numbers 56, 33, and 65; and from the fractions $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$, we may derive the whole numbers 72, 65, and 97; all which ſets of whole numbers will expreſs the ſides of right-angled triangles ſimilar to the three former triangles, of which the ſides were expreſſed by the fractions $\frac{20}{3}$, $\frac{21}{3}$, and $\frac{29}{3}$, and the fractions $\frac{56}{3}$, $\frac{33}{3}$, and $\frac{65}{3}$, and the fractions $\frac{72}{5}$, $\frac{65}{5}$, and $\frac{97}{5}$.

Art. 4. And theſe whole numbers might have been obtained at once by computing only the numerators of the three general fractions $\frac{nn - ee}{2e}$, $\frac{2en}{2e}$, and $\frac{nn + ee}{2e}$, to wit, the expreſſions $nn - ee$, $2en$, and $nn + ee$, which are the products of the multiplication of the ſaid three fractions into their common denominator $2e$. For then we ſhould have found, in the firſt example, in which e is $= 1$ and n is $= 2$, that the ſaid expreſſions $nn - ee$, $2en$, and $nn + ee$, would have been equal to ($4 - 1$, $2 \times 1 \times 2$, and $4 + 1$, or) 3, 4, and 5; and, in the ſecond example, in which e was $= 2$, and $n = 3$, we ſhould have had $nn - ee$, $2en$, and $nn + ee$, equal to ($9 - 4$, $2 \times 2 \times 3$, and $9 + 4$, or) 5, 12, 5, 12, and 13; and, in the 3d example, in which e was $= 3$, and $n = 5$, we should have had $nn - ee$, $2en$, and $nn + ee$, equal to $(25 - 9, 2 \times 3 \times 5$, and $25 + 9$, or) 16, 30, and 34, which, when divided by 2, become 8, 15, and 17; and in the 4th example, in which e was $= 3$, and $n = 7$, we should have had $nn - ee$, $2en$, and $nn + ee$, equal to $(49 - 9, 2 \times 3 \times 7$, and $49 + 9$, or) 40, 42, and 58, which, when divided by 2, become 20, 21, and 29; and, in the 5th example, in which e is $= 3$, and n is $= 11$, we should have had $nn - ee$, $2en$, and $nn + ee$, equal to $(121 - 9, 2 \times 3 \times 11$, and $121 + 9$, or) 112, 66, and 130, which, when divided by 2, become 56, 33, and 65; and in the 6th and last example, in which e was $= 5$, and n was $= 13$, we should have had $nn - ee$, $2en$, and $nn + ee$, equal to $(169 - 25, 2 \times 5 \times 13$, and $169 + 25$, or) 144, 130, and 194, which, when divided by 2, become 72, 65, and 97. And thus we should have obtained the six foregoing sets of whole numbers to express the sides of different right-angled triangles, to wit, 1st, the numbers 3, 4, and 5; 2dly, the numbers 5, 12, and 13; 3dly, the numbers 8, 15, and 17; 4thly, the numbers 20, 21, and 29; 5thly, the numbers 56, 33, and 65; and, 6thly, the numbers 72, 65, and 97.

A SCHOLIUM.

Art. 5. It may be observed, that in the four first of the foregoing six sets of numbers, which express the sides of right-angled triangles, to wit, in the numbers 3, 4, and 5, and in the numbers 5, 12, and 13, and in the numbers 8, 15, and 17, and in the numbers 20, 21, and 29, the first number of each set is less than the second; but in the fifth and sixth sets of those numbers, to wit, in the numbers 56, 33, and 65, and in the numbers 72, 65, and 97, the

first

first number of each set is greater than the second. Now this depends upon the proportion in which the number n (which is always greater than e,) exceeds the number e. For, if n were not a number, but a quantity incommensurable to 1, which bore the same proportion to the number e as 1 bears to $\sqrt{2} - 1$, or to the excess of $\sqrt{2}$ above 1, or as the side of a square bears to the excess of its diagonal above its side, the general expression $nn - ee$, from which the first terms of all these sets of numbers are derived, would be exactly equal to the general expression $2en$, from which the second terms of the said sets of numbers are derived: and, when the proportion of n to e is less than that of 1 to $\sqrt{2} - 1$, or n is less than $e \times \frac{1}{\sqrt{2-1}}$, the general expression $nn - ee$ will be less than $2en$: and, when the proportion of n to e is greater than the said proportion of 1 to $\sqrt{2} - 1$, or n is greater than $e \times \frac{1}{\sqrt{2-1}}$, the general expression $nn - ee$ will be greater than $2en$. These things may be demonstrated in the manner following.

Art. 6. In the 1st place, if n is $= e \times \frac{1}{\sqrt{2-1}}$, we shall have $nn = ee \times \frac{1}{\overline{\sqrt{2-1}}|^2} = \frac{ee}{2 - 2\sqrt{2} + 1}$, and $nn - ee$ $\left(= \frac{ee}{2-2\sqrt{2}+1} - ee = \frac{ee}{2-2\sqrt{2}+1} - \frac{\overline{2-2\sqrt{2}+1} \times ee}{2-2\sqrt{2}+1} = \frac{ee}{2-2\sqrt{2}+1} - \frac{\overline{2ee - 2\sqrt{2} \times ee + ee}}{2-2\sqrt{2}+1} = \frac{ee}{2-2\sqrt{2}+1} - \frac{\overline{3ee - 2\sqrt{2} \times ee}}{2-2\sqrt{2}+1} = \frac{ee}{2-2\sqrt{2}+1} \frac{-3ee + 2\sqrt{2} \times ee}{2-2\sqrt{2}+1} = + \frac{2\sqrt{2} \times ee - 2ee}{2-2\sqrt{2}+1} = \frac{2ee \times \overline{\sqrt{2}-1}}{2-2\sqrt{2}+1}\right) = \frac{2ee}{\sqrt{2}-1}$; and we shall also have $2en$ $\left(= 2e \times e \times \frac{1}{\sqrt{2}-1}\right) = \frac{2ee}{\sqrt{2}-1}$. And consequently $nn - ee$ will in this case be equal to $2en$. Q. E. D.

 2dly,

2dly, If n is leſs than $e \times \frac{1}{\sqrt{2} - 1}$, the compound quantity $nn - ee$ will be leſs than $2en$.

For, if we ſuppoſe n, from being equal to $e \times \frac{1}{\sqrt{2} - 1}$, to become leſs than that quantity, but ſtill to be greater than e, and the decrement of n, or its difference from its firſt value, to be denoted by the letter d, it is evident that while n is decreaſing from n to $n - d$, the compound quantity $nn - ee$ will decreaſe from $nn - ee$ to $\overline{n - d}|^2 - ee$, that is, to $nn - 2nd + dd - ee$, or to $nn - ee - 2nd + dd$, or to $nn - ee - \overline{2nd - dd}$, or will be leſs than it was before by the quantity $2nd - dd$; and in the ſame time the quantity $2en$ will decreaſe from its firſt value, $2en$, (which was equal to $nn - ee$) to $2e \times \overline{n - d}$, or $2en - 2ed$, or will be leſs than it was before by the quantity $2ed$. Now, becauſe $n - d$ is greater than e, it follows that n muſt be greater than $e + d$, and conſequently that $n - e$ muſt be greater than d. Therefore $2d \times \overline{n - e}$ will be greater than $2d \times d$, or $2nd - 2ed$ will be greater than $2dd$, and conſequently $2nd$ will be greater than $2dd + 2ed$, and $2nd - dd$ will be greater than $dd + 2ed$. Therefore, *à fortiori*, $2nd - dd$ will be greater than $2ed$; that is, the decrement of the quantity $nn - ee$ while n decreaſes from n to $n - d$, will be greater than the decrement of the quantity $2en$ in the ſame time: and conſequently the quantity $nn - ee - \overline{2nd - dd}$, to which $nn - ee$ will have decreaſed, while n was decreaſing from n to $n - d$, will be leſs than the quantity $2en - 2ed$, to which the quantity $2en$ (which was at firſt equal to $nn - ee$,) will have decreaſed in the ſame time; or, if n is of any magnitude leſs than $e \times \frac{1}{\sqrt{2} - 1}$, but greater than e, the quantity $nn - ee$ will be leſs than the quantity $2en$. Q. E. D.

And, 3dly, if n is greater than $e \times \frac{1}{\sqrt{2 - 1}}$, the compound quantity $nn - ee$ will be greater than $2en$.

For,

For, if we ſuppoſe n, from being equal to $e \times \frac{1}{\sqrt{2}-1}$, to become greater than that quantity, and the increment of n, or its difference from its former value, to be denoted by the letter d, it is evident that, while n is increaſing from n to $n + d$, the compound quantity $nn - ee$ will increaſe from $nn - ee$ to $\overline{n+d}|^2 - ee$, that is, to $nn + 2nd + dd - ee$, or to $nn - ee + 2nd + dd$, or it will be greater than it was before by the quantity $2nd + dd$; and in the ſame time the quantity $2en$ will increaſe from its firſt value, $2en$, (which was equal to $nn - ee$,) to $2e \times \overline{n+d}$, or $2en + 2ed$, or will be greater than it was before by the quantity $2ed$. Now, becauſe n is greater than e, it follows that $2nd$ muſt be greater than $2ed$; and conſequently, *à fortiori*, $2nd + dd$ will be greater than $2ed$; that is, the increment of $nn - ee$ during the increaſe of n from n to $n + d$ will be greater than the contemporary increment of $2en$. Therefore the quantity $nn - ee + 2nd + dd$, to which $nn - ee$ will have increaſed while n increaſed from n to $n + d$, will be greater than the quantity $2en + 2ed$, to which the quantity $2en$ (which was at firſt equal to $nn - ee$,) will have increaſed in the ſame time; or, if n is of any magnitude greater than $e \times \frac{1}{\sqrt{2}-1}$, the quantity $nn - ee$ will be greater than the quantity $2en$. Q. E. D.

Art. 7. If we take two numbers for e and n that are nearly in the proportion of $\sqrt{2} - 1$ and 1, we ſhall find that $nn - ee$ will be very nearly equal to $2en$. Now $\sqrt{2}$ is $=$ 1.414 &c. Therefore $\sqrt{2} - 1$ is $=$ 1.414 &c $-$ 1 $=$ 0.414 &c, and $\sqrt{2} - 1$ is to 1 pretty nearly in the proportion of 0.414 to 1, or of 414 to 1000, or of 207 to 500. Therefore, if we ſuppoſe e to be $=$ 207, and n to be $=$ 500, the value of $nn - ee$ ought to be nearly equal to that of $2en$. And ſo we ſhall find them to be. For, upon theſe ſuppoſitions, we ſhall have ee ($= \overline{207}|^2$) $=$ 42,849, and nn ($= \overline{500}|^2$) $=$ 250,000, and $nn - ee$ ($=$ 250,000 $-$ 42,849) $=$ 207,151. And $2en$ will be ($=$

2 × 207 × 500 = 207 × 1000) = 207,000; which is very nearly equal to 207,151, or *nn* — *ee*.

Art. 8. And, upon theſe ſuppoſitions, *nn* + *ee* will be (= 250,000 + 42,849) = 292,849; which gives us a ſeventh ſet of numbers that expreſs the ſides of a right-angled triangle, to wit, the numbers 207,151, 207,000, and 292,849. And accordingly we ſhall find that the ſquare of the number 292,849, which repreſents the hypotenuſe of the triangle, will be equal to the ſum of the ſquares of the other two numbers 207,151 and 207,000, which repreſent the ſides that contain the right angle. For the ſquare of 292,849 is 85,760,536,801, and the ſquares of 207,151 and 207,000, are 42,911,536,801 and 42,849,000,000; and 85,760,536,801 is = 42,911,536,801 + 42,849,000,000.

Art. 9. If we ſuppoſe *e* to be = 2, and *n* to be = 5, we ſhall have *nn* — *ee* (= 25 — 4) = 21, and 2*en* (= 2 × 2 × 5) = 20, and *nn* + *ee* (= 25 + 4) = 29. Therefore 21, 20, and 29, will be three numbers that will expreſs the lengths of the three ſides of a right-angled triangle. And theſe numbers, we may obſerve, are the ſame with the three numbers 20, 21, and 29, obtained above in art. 4, by ſuppoſing *e* to be equal to 3, and *n* to be equal to 7, excepting that the order of the two firſt numbers 20 and 21 is different in the two ſets, 20 being the firſt number in the firſt ſet, 20, 21, and 29, and being the ſecond number in the ſecond ſet, 21, 20, and 29; the reaſon of which is, that 20 is derived from the firſt general expreſſion *nn* — *ee* in the firſt ſet of numbers, 20, 21, and 29, and it is derived from the ſecond general quantity 2*en* in the ſecond ſet of numbers, 21, 20, and 29. This, however, has an odd appearance, that, when the original numbers *n* and *e*, from which the general expreſſions *nn* — *ee*, 2*en*, and *nn* + *ee* are derived, are made to bear different proportions to each other (for the proportion of 5 to 2 is greater than the proportion of 7 to 3, being equal to that of 7 to 2.8, or of 70 to 28,) the three numbers obtained by means of of thoſe general expreſſions ſhould ſtill be the ſame, though placed in a different order: and therefore it may not be amiſs to inquire a little further into it.

Art. 10. In order therefore that the reaſon of this ſeeming irregularity may appear the more clearly, we will recur to the obſervation made above in art. 5, concerning the change in the proportion of the general expreſſion $nn - ee$ to the general expreſſion $2en$, when the proportion of n to e, from being at firſt a leſs proportion of majority than that of 1 to $\surd 2 - 1$, or of 1 to 0.414, &c, becomes equal to, and greater than, the ſaid ratio; to wit, that, when the ratio of n to e is leſs than the ratio of 1 to 0.414, &c, the quantity $nn - ee$ is leſs than the quantity $2en$; and that, when the ratio of n to e is equal to the ratio of 1 to 0.414, &c, the quantity $nn - ee$ is equal to the quantity $2en$; and that, when the ratio of n to e is greater than the ratio of 1 to 0.414, &c, the quantity $nn - ee$ is greater than the quantity $2en$. And to this obſervation we muſt add that, if the ratio of n to e, after having been equal to the ratio of 1 to 0.414, &c, is ſuppoſed to increaſe gradually *ad infinitum*, the ratio of $nn - ee$ to $2en$ will increaſe gradually at the ſame time *ad infinitum*, or ſo as to become greater than any aſſigned ratio whatſoever. For the ratio of $nn - ee$ to $2en$ is equal to the ratio of $\frac{nn - ee}{2en}$ to $(\frac{2en}{2en}$, or) 1, or to the ratio of $\frac{nn}{2en} - \frac{ee}{2en}$ to 1, or to the ratio of $\frac{n}{2e} - \frac{e}{2n}$ to 1, which evidently increaſes *ad infinitum*, while the ratio of n to e increaſes *ad infinitum*. Therefore while the ratio of n to e increaſes, from being equal to the ratio of 1 to 0.414, &c, *ad infinitum*, the ratio of $nn - ee$ to $2en$ will increaſe gradually from a ratio of equality *ad infinitum*, and conſequently will become ſucceſſively equal to all ratios of majority whatſoever. Therefore, if the ratio of n to e is at one time taken equal to the ratio of 7 to 3, (which is leſs than the ratio of 7 to 2.898, &c, or of 7 to 7×0.414, &c, or of 1 to 0.414, &c,) and is afterwards ſuppoſed to increaſe

increaſe gradually till it becomes equal to the ſaid ratio of 1 to 0.414, &c, and then to increaſe further *ad infinitum*; the ratio of the compound quantity *nn* — *ee* to the quantity 2*en* (which is equal to the ratio of 20 to 21, when the ratio of *n* to *e* is equal to the ratio of 7 to 3,) will firſt become a ratio of equality, to wit, when the ratio of *n* to *e* becomes equal to the ratio of 1 to 0.414, &c, and afterwards will increaſe continually from being a ratio of equality, (which may be conſidered as an infinitely ſmall ratio of majority, and is uſually ſo conſidered by writers who treat of the magnitudes and meaſures of ratios,) till it becomes ſucceſſively equal to every ratio of majority whatſoever. It therefore muſt at one point of time during its ſaid increaſe become equal to the ratio of 21 to 20; or, there will be a certain ratio of majority, greater than that of 1 to 0.414, &c, to which when the ratio of *n* to *e* ſhall have become equal, the ratio of *nn* — *ee* to 2*en* will be equal to the ratio of 21 to 20. And this ratio of majority is that of 5 to 2, as has been ſhewn in the foregoing, or 9th, article.

Art. 11. I will juſt add one more example of the foregoing method of finding three whole numbers that ſhall expreſs the lengths of the ſides of a right-angled triangle.

Let *e* be = 5, and *n* be = 17. And we ſhall have *ee* = 25, and *nn* = 289, and *nn* — *ee* (= 289 — 25) = 264, and 2*en* (= 2 × 5 × 17 = 10 × 17) = 170, and *nn* + *ee* (= 289 + 25) = 314. Therefore 264, 170, and 314, or (dividing all theſe numbers by 2,) 132, 85, and 157, will be three whole numbers that will expreſs the three ſides of a right-angled triangle.

And accordingly we ſhall find that the ſquare of the number 157, which repreſents the hypotenuſe, or line ſubtending the right angle, will be equal to the ſum of the ſquares of the two numbers 132 and 85, which repreſent the ſides that contain the right angle. For the ſquare of 157 is 24,649, and the ſquares of 132 and 85 are 17,424 and 7225; and 24,649 is = 17,424 + 7225.

We

We have therefore now found the nine following ſets of whole numbers for expreſſing the three ſides of different right-angled triangles; to wit,

1ſt, The whole numbers 3, 4, and 5;

2dly, The whole numbers 5, 12, and 13;

3dly, The whole numbers 8, 15, and 17;

4thly, The whole numbers 20, 21, and 29;

5thly, The whole numbers 56, 33, and 65;

6thly, The whole numbers 72, 65, and 97;

7thly, The whole numbers 207,151, 207,000, and 292,849;

8thly, The whole numbers 21, 20, and 29; and

9thly, The whole numbers 132, 85, and 157.

And we may eaſily find as many more ſets of ſuch numbers as we pleaſe, by ſubſtituting different numbers for e and n, or for either of them, in the three general expreſſions $nn - ee$, $2en$, and $nn + ee$.

Art. 12. The foregoing ſubject may alſo be treated in a ſomewhat different manner, by ſolving the following Problem.

A PROBLEM.

To divide a given ſquare number into two other ſquare numbers, either whole numbers, or fractions, or mixt numbers.

SOLUTION.

SOLUTION.

Let the given square number that is to be so divided, be denoted by the letters aa, and let xx be one of the two square numbers that are sought, and yy be the other.

Then, since the two numbers sought are together to be equal to the given number, we shall, in the first place, have $xx + yy = aa$.

Now, since $xx + yy$ is $= aa$, it follows that yy will be $= aa - xx$. But $aa - xx$ is $= \overline{a + x} \times \overline{a - x}$. Therefore yy will be $= \overline{a + x} \times \overline{a - x}$. Therefore y will be a geometrical mean proportional between $a + x$ and $a - x$, and consequently will be less than $a + x$. Let the proportion of $a + x$ to y be that of the two numbers m and n, of which m is the greater. Then will y be $= \frac{n}{m} \times \overline{a + x}$, and yy will be $= \frac{nn}{mm} \times \overline{a + x}|^2$ $(= \frac{nn}{mm} \times \overline{aa + 2ax + xx})$ $= \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax + \frac{nn}{mm} \times xx$. Therefore $xx + yy$ will be $= xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax + \frac{nn}{mm} \times xx$ $(= \frac{mm}{mm} \times xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax + \frac{nn}{mm} \times xx)$ $= \frac{mm + nn}{mm} \times xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax$. But $xx + yy$ is $= aa$. Therefore $\frac{mm + nn}{mm} \times xx + \frac{nn}{mm} \times aa + \frac{nn}{mm} \times 2ax$ will also be $= aa$, and consequently $= \frac{mm}{mm} \times aa$. Therefore $\frac{mm + nn}{mm} \times xx + \frac{nn}{mm} \times 2ax$ will be $= \frac{mm}{mm} \times aa - \frac{nn}{mm} \times aa = \frac{mm - nn}{mm} \times aa$, and con-

sequently

ſequently (multiplying all the terms by mm,) $\overline{mm + nn} \times xx + nn \times 2ax$ will be $= \overline{mm - nn} \times aa$, and (dividing all the terms by $mm + nn$,) $xx + \frac{nn}{mm + nn} \times 2ax$ will be $= \frac{mm - nn}{mm + nn} \times aa$. Therefore (adding $\frac{n^4}{\overline{mm + nn}|^2} \times aa$ to both ſides of the equation,) we ſhall have $xx + \frac{nn}{mm + nn} \times 2ax + \frac{n^4}{\overline{mm + nn}|^2} \times aa = \frac{mm - nn}{mm + nn} \times aa + \frac{n^4}{\overline{mm + nn}|^2} \times aa = \frac{\overline{mm - nn} \times \overline{mm + nn}}{mm + nn \times mm + nn} \times aa + \frac{n^4}{\overline{mm + nn}|^2} \times aa = \frac{m^4 - n^4}{\overline{mm + nn}|^2} \times aa + \frac{n^4}{\overline{mm + nn}|^2} \times aa = \frac{m^4}{\overline{mm + nn}|^2} \times aa$. Therefore (extracting the ſquare-roots of both ſides,) we ſhall have $x + \frac{nn}{mm + nn} \times a = \frac{mm}{mm + nn} \times a$, and conſequently $x = \frac{mm}{mm + nn} \times a - \frac{nn}{mm + nn} \times a = \frac{mm - nn}{mm + nn} \times a$. Therefore $a + x$ will be $= a + \frac{mm - nn}{mm + nn} \times a$ ($= \frac{mm + nn}{mm + nn} \times a + \frac{mm - nn}{mm + nn} \times a$) $= \frac{2mm}{mm + nn} \times a$; and y, or $\frac{n}{m} \times a + x$, will be $= \frac{n}{m} \times \frac{2mm}{mm + nn} \times a = \frac{2mn}{mm + nn} \times a$, that is, x, or the root, or ſide, of the firſt of the two ſquares ſought, to wit, xx and yy, will be equal to $\frac{mm - nn}{mm + nn} \times a$, and y, or the root, or ſide, of the latter of the ſaid two ſquares, will be equal to $\frac{2mn}{mm + nn} \times a$; and conſequently xx, or the firſt of the ſaid two ſquares itſelf, will be equal to $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$, and yy, or the latter of the ſaid two ſquares itſelf, will be equal to $\frac{\overline{2mn}^2}{\overline{mm+nn}^2} \times aa$, or $\frac{4m^2n^2}{\overline{mm+nn}^2} \times aa$; or, if m and n be any two numbers whatſoever, of which m is the greater, $\frac{\overline{mm-nn}^2}{\overline{mm+nn}^2} \times aa$ and $\frac{4m^2n^2}{\overline{mm+nn}^2} \times aa$ will be two ſquare numbers that will together be equal to the original ſquare number aa. Q. E. I.

Art. 13. That theſe two ſquare numbers $\frac{\overline{mm-nn}^2}{\overline{mm+nn}^2} \times aa$ and $\frac{4m^2n^2}{\overline{mm+nn}^2} \times aa$, will together be equal to the original ſquare number aa, will be evident by adding them together. For $\frac{\overline{mm-nn}^2}{\overline{mm+nn}^2} \times aa + \frac{4m^2n^2}{\overline{mm+nn}^2} \times aa$ are $= \frac{m^4 - 2m^2n^2 + n^4}{m^4 + 2m^2n^2 + n^4} \times aa + \frac{4m^2n^2}{m^4 + 2m^2n^2 + n^4} \times aa = \frac{m^4 + 2m^2n^2 + n^4}{m^4 + 2m^2n^2 + n^4} \times aa = aa$.

Q. E. D.

This ſolution of the foregoing Problem ſeems to me more eaſy and natural than that of Dr. Saunderſon in his Algebra, Vol. 2, page 366, *et ſequentibus*.

Examples of the foregoing Solution.

Art. 14. Let the given ſquare number aa be 25, and let n be 1, and m be 2.

Then will nn be $= 1$, and mm be $= 4$, and $mm - nn$ will be $(= 4 - 1) = 3$, and $mm + nn$ will be $(= 4 + 1) = 5$, and conſequently $\overline{mm - nn}^2$ will be $= 9$, and $\overline{mm + nn}^2$ will be $= 25$, and $4m^2n^2$ will be $(= 4 \times 4 \times 1) = 16$,

$= 16$, and $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{9}{25} \times 25 = 9$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{16}{25} \times 25 = 16$. Therefore 9 and 16 will be two ſquare numbers that will together be equal to the given ſquare number 25. And it is evident that 9 and 16 are equal to 25.

Secondly, let aa be $= 25$, and $n = 1$, and $m = 3$.

Then will nn be $= 1$, and mm be $= 9$, and conſequently $mm - nn$ will be $(= 9 - 1) = 8$, and $mm + nn$ will be $= 9 + 1 = 10$, and $4m^2n^2$ will be $(= 4 \times 9 \times 1) = 36$. Therefore $\overline{mm - nn}|^2$ will be $= 64$, and $\overline{mm + nn}|^2$ will be $= 100$, and $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{64}{100} \times aa = \frac{64}{100} \times 25 = \frac{16}{25} \times 25 = 16$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{36}{100} \times aa = \frac{9}{25} \times aa = \frac{9}{25} \times 25 = 9$. Therefore 16 and 9 will be two ſquare numbers that will together be equal to the given ſquare number 25. And it is evident that they are equal to 25.

Theſe two ſquare numbers 16 and 9 are the ſame with the two former ſquare numbers 9 and 16, derived from the ſuppoſition that m was $= 3$, except in the order of their poſition.

Thirdly, let aa be $= 25$, and n be $= 1$, and m be $= 4$.

Then we ſhall have $nn = 1$, and $mm = 16$, and conſequently $mm - nn$ $(= 16 - 1) = 15$, and $mm + nn$ $(= 16 + 1) = 17$, and $\overline{mm - nn}|^2$ $(= \overline{15}|^2) = 225$, and $\overline{mm + nn}|^2$ $(= \overline{17}|^2) = 289$, and $4m^2n^2$ $(= 4 \times 16 \times 1) = 64$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{225}{289} \times aa)$

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$= \frac{225}{289} \times 25$, and $\frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$ will be $(= \frac{64}{289} \times aa)$ $= \frac{64}{289} \times 25$. Therefore $\frac{225}{289} \times 25$ and $\frac{64}{289} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And it is evident that theſe two ſquare numbers are equal to 25: for $\frac{225}{289} \times 25 + \frac{64}{289} \times 25$ are $= \frac{225+64}{289} \times 25 = \frac{289}{289} \times 25 = 25$.

If we multiply theſe three numbers $\frac{225}{289} \times 25$, $\frac{64}{289} \times 25$, and 25, by 289, we ſhall thereby obtain the three following whole numbers, to wit, 225×25, 64×25, and 289×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three laſt numbers by 25, we ſhall obtain the three following leſſer whole numbers, to wit, 225, 64, and 289, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter.

Fourthly, let aa be $= 25$, and n be $= 1$, and m be $= 5$.

Then we ſhall have $nn = 1$, and $mm = 25$, and conſequently $mm - nn = 24$, and $mm + nn = 26$, and $\overline{mm-nn}|^2$ $(= \overline{24}|^2 = 576$, and $\overline{mm+nn}|^2$ $(= \overline{26}|^2) = 676$, and $4m^2n^2$ $(= 4 \times 25 \times 1) = 100$. Therefore $\frac{\overline{mm-nn}|^2}{\overline{mm+nn}|^2} \times aa$ will be $(= \frac{576}{676} \times aa) = \frac{576}{676} \times 25$, and $\frac{4m^2n^2}{\overline{mm+nn}|^2} \times aa$ will be $(= \frac{100}{676} \times aa) = \frac{100}{676} \times 25$. Therefore $\frac{576}{676} \times 25$ and $\frac{100}{676} \times 25$ will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And accord-

accordingly it is evident that theſe two ſquare numbers are equal to 25: for $\frac{576}{676} \times 25 + \frac{100}{676} \times 25$ are $= \frac{576 + 100}{676} \times 25 = \frac{676}{676} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{576}{676} \times 25$, $\frac{100}{676} \times 25$, and 25, by 676, we ſhall thereby obtain the three following whole numbers, to wit, 576×25, 100×25, and 676×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three laſt numbers by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 576, 100, and 676, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe laſt numbers, 576, 100, and 676, by 4, we ſhall thereby obtain the three following ſtill leſſer whole numbers, to wit, 144, 25, and 169, which will likewiſe be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter.

Art. 15. In the 5th place, let aa be, as before, $= 25$, and let n be $= 2$, and m be $= 3$.

Then we ſhall have $nn = 4$, and $mm = 9$, and conſequently $mm - nn$ $(= 9 - 4) = 5$, and $mm + nn$ $(= 9 + 4) = 13$, and $\overline{mm - nn}|^2 = 25$, and $\overline{mm + nn}|^2 = 169$, and $4m^2n^2$ $(= 4 \times 9 \times 4) = 144$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{25}{169} \times aa) = \frac{25}{169} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{144}{169} \times aa) = \frac{144}{169} \times 25$. Therefore $\frac{25}{169} \times 25$ and $\frac{144}{169} \times 25$ will be two ſquare numbers that will, together, be equal to 25. And accordingly it is evident that theſe two ſquare numbers $\frac{25}{169} \times 25$ and $\frac{144}{169}$ $\times 25$

$\times 25$ are equal to 25. For they are equal to $\frac{25+144}{169} \times 25 = \frac{169}{169} \times 25 = 25$.

If we multiply these three numbers, $\frac{25}{169} \times 25$, $\frac{144}{169} \times 25$, and 25, by 169, we shall thereby obtain the three following whole numbers, to wit, 25×25, 144×25, and 169×25, which will be, all of them, square numbers; and of which the two former will, together, be equal to the latter. And, if we divide these three last numbers 25×25, 144×25, and 169×25, by 25, we shall thereby obtain the three following lesser whole numbers, to wit, 25, 144, and 169, which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter.

Sixthly, let aa be, as before, $= 25$, and let n be $= 2$, and m be $= 5$.

Then we shall have $nn = 4$, and $mm = 25$, and consequently $mm - nn$ $(= 25 - 4) = 21$, and $mm + nn$ $(= 25 + 4) = 29$, and $\overline{mm - nn}|^2$ $(= \overline{21}|^2) = 441$, and $\overline{mm + nn}|^2$ $(= \overline{29}|^2) = 841$, and $4m^2n^2$ $(= 4 \times 25 \times 4) = 400$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{441}{841} \times aa) = \frac{441}{841} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{400}{841} \times 25$. Therefore $\frac{441}{841} \times 25$ and $\frac{400}{841} \times 25$, will be two square numbers that will, together, be equal to the given square number 25. And so we shall find them to be: for $\frac{441}{841} \times 25 + \frac{400}{841} \times 25$ are $= \frac{441 + 400}{841} \times 25 = \frac{841}{841} \times 25 = 25$.

If

If we multiply these three numbers, $\frac{441}{841} \times 25$, $\frac{400}{841} \times 25$, and 25, by 841; we shall thereby obtain the three following whole numbers, to wit, 441×25, 400×25, and 841×25, which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter. And, if we divide these three whole numbers by 25, we shall obtain the three following lesser whole numbers, to wit, 441, 400, and 841, which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter.

Seventhly, let aa be, as before, $= 25$, and let n be $= 2$, and m be $= 7$.

Then we shall have $nn = 4$, and $mm = 49$, and consequently $mm - nn$ $(= 49 - 4) = 45$, and $mm + nn$ $(= 49 + 4) = 53$, and $\overline{mm - nn}|^2$ $(= \overline{45}|^2) = 2025$, and $\overline{mm + nn}|^2$ $(= \overline{53}|^2) = 2809$, and $4m^2n^2$ $(= 4 \times 49 \times 4) = 784$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{2025}{2809} \times aa) = \frac{2025}{2809} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $= \frac{784}{2809} \times 25$. Therefore $\frac{2025}{2809} \times 25$ and $\frac{784}{2809} \times 25$, will be two square numbers that will, together, be equal to the given square number 25. And accordingly it is evident that these two square numbers $\frac{2025}{2809} \times 25$, and $\frac{784}{2809} \times 25$, are equal to 25. For they are equal to $\frac{2025 + 784}{2809} \times 25 = \frac{2809}{2809} \times 25 = 25$.

If we multiply these three numbers, $\frac{2025}{2809} \times 25$, $\frac{784}{2809} \times 25$, and 25 by 2809, we shall thereby obtain the three following whole numbers, to wit, 2025×25, 784×25, and 2809×25, which will be, all of them, square numbers, and

and of which the two former will, together, be equal to the latter. And, if we divide these three whole numbers by 25, we shall thereby obtain the three following lesser whole numbers, to wit, 2025, 784, and 2809, which will, all of them, be square numbers, and of which the two former will, together, be equal to the latter.

Art. 16. In the 8th place, let aa be, as before, equal to 25, and let n be $= 3$, and m be $= 5$.

Then we shall have $nn = 9$, and $mm = 25$, and consequently $mm - nn$ $(= 25 - 9) = 16$, and $mm + nn$ $(= 25 + 9) = 34$, and $\overline{mm - nn}|^2$ $(= \overline{16}|^2) = 256$, and $\overline{mm + nn}|^2$ $(= \overline{34}|^2) = 1156$, and $4m^2n^2$ $(= 4 \times 25 \times 9 = 100 \times 9) = 900$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{256}{1156} \times aa) = \frac{256}{1156} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{900}{1156} \times aa) = \frac{900}{1156} \times 25$. Therefore $\frac{256}{1156} \times 25$, and $\frac{900}{1156} \times 25$, will be two square numbers that will, together, be equal to the given square number 25. And accordingly it is evident that these two square numbers $\frac{256}{1156} \times 25$, and $\frac{900}{1156} \times 25$, are equal to 25. For they are $= \frac{256 + 900}{1156} \times 25 = \frac{1156}{1156} \times 25 = 25$.

If we multiply these three square numbers, $\frac{256}{1156} \times 25$, $\frac{900}{1156} \times 25$, and 25, by 1156, we shall thereby obtain the three following whole numbers, to wit, 256×25, 900×25, and 1156×25, which will be, all of them, square numbers, and of which the two former will, together, be equal to the latter. And, if we divide these three whole numbers by 25, we shall thereby obtain the three following lesser,

leſſer whole numbers, to wit, 256, 900, and 1156, which are, all of them, ſquare numbers, and of which the two former are, together, equal to the latter.

Ninthly, let aa be, as before, $= 25$, and let n be $= 3$, and m be $= 7$.

Then we ſhall have $nn = 9$, and $mm = 49$, and conſequently $mm - nn$ $(= 49 - 9) = 40$, and $mm + nn$ $(= 49 + 9) = 58$, and $\overline{mm - nn}|^2$ $(= 40^2) = 1600$, and $\overline{mm + nn}|^2$ $(= 58^2) = 3364$, and $4m^2n^2$ $(= 4 \times 49 \times 9) = 1764$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{1600}{3364} \times aa) = \frac{1600}{3364} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{1764}{3364} \times aa) = \frac{1764}{3364} \times 25$. Therefore $\frac{1600}{3364} \times 25$, and $\frac{1764}{3364} \times 25$, will be two ſquare numbers that, together, will be equal to the given ſquare number 25. And accordingly it is evident that theſe two numbers are equal to 25. For $\frac{1600}{3364} \times 25 + \frac{1764}{3364} \times 25$, are $= \frac{1600 + 1764}{3364} \times 25 = \frac{3364}{3364} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{1600}{3364} \times 25$, $\frac{1764}{3364} \times 25$, and 25, by 3364, we ſhall thereby obtain the three whole numbers 1600×25, 1764×25, and 3364×25, which will be, all of them, ſquare numbers, and of which the two former will, together, be equal to the latter. And, if we divide theſe three whole numbers by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 1600, 1764, and 3364, which are, all of them, ſquare numbers, and of which the two former are equal to the latter.

 Tenthly,

Tenthly, let aa be, as before, $= 25$, and let n be $= 3$, and m be $= 11$.

Then we ſhall have $nn = 9$, and $mm = 121$, and conſequently $mm - nn$ $(= 121 - 9) = 112$, and $mm + nn$ $(= 121 + 9) = 130$, and $\overline{mm - nn}|^2$ $(= \overline{112}|^2) = 12,544$, and $\overline{mm + nn}|^2$ $(= \overline{130}|^2) = 16,900$, and $4m^2n^2$ $(= 4 \times 121 \times 9 = 484 \times 9) = 4356$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{12,544}{16,900} \times aa) = \frac{12,544}{16,900} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{4356}{16,900} \times aa) = \frac{4356}{16,900} \times 25$. Therefore $\frac{12,544}{16,900} \times 25$, and $\frac{4356}{16,900} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And accordingly it will eaſily appear that theſe two numbers are equal to 25. For $\frac{12,544}{16,900} \times 25 + \frac{4356}{16,900} \times 25$ are $= \frac{12,544 + 4356}{16,900} \times 25 = \frac{16,900}{16,900} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{12,544}{16,900} \times 25$, $\frac{4356}{16,900} \times 25$, and 25, by 16,900, we ſhall thereby obtain the three following whole numbers, to wit, $12,544 \times 25$, 4356×25, and $16,900 \times 25$, which are, all of them, ſquare numbers, and of which the two former are, together, equal to the latter. And, if we divide theſe whole numbers by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 12,544, 4356, and 16,900, which are, all of them, ſquare numbers, (to wit, the ſquares of the numbers 112, 66, and 130,) and of which the two former are, together, equal to the latter.

Art. 17. In the 11th place, let aa be equal, as before, to 25, and let n be $= 5$, and m be $= 7$.

Then

Then we ſhall have $nn = 25$, and $mm = 49$, and conſequently $mm - nn$ $(= 49 - 25) = 24$, and $mm + nn$ $(= 49 + 25) = 74$, and $\overline{mm - nn}|^2$ $(= \overline{24}|^2) = 576$; and $\overline{mm + nn}|^2$ $(= \overline{74}|^2) = 5476$, and $4m^2n^2$ $(= 4 \times 49 \times 25 = 100 \times 49) = 4900$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{576}{5476} \times aa) = \frac{576}{5476} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be $(= \frac{4900}{5476} \times aa) = \frac{4900}{5476} \times 25$. Therefore $\frac{576}{5476} \times 25$, and $\frac{4900}{5476} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And accordingly, if we add theſe numbers together, we ſhall find them to be $(= \frac{576 + 4900}{5476} \times 25 = \frac{5476}{5476} \times 25) = 25$.

If we multiply theſe three numbers, $\frac{576}{5476} \times 25$, $\frac{4900}{5476} \times 25$, and 25, by 5476, we ſhall thereby obtain the three following whole numbers, to wit, 576×25, 4900×25, and 5476×25, which are, all of them, ſquare numbers, and of which the two former are, together, equal to the latter. And, if we divide theſe whole numbers by 25, we ſhall thereby obtain the three following leſſer whole numbers, to wit, 576, 4900, and 5476, which are alſo, all of them, ſquare numbers, and of which the two former are, together, equal to the latter.

Twelfthly, let aa be, as before, $= 25$, and let n be $= 5$, and m be $= 11$.

Then we ſhall have $nn = 25$, and $mm = 121$, and conſequently $mm - nn$ $(= 121 - 25) = 96$, and $mm + nn$ $(= 121 + 25) = 146$, and $\overline{mm - nn}|^2$ $(= \overline{96}|^2) = 9216$,

 and

and $\overline{mm + nn}|^2$ ($= \overline{146}|^2$) $= 21,316$, and $4m^2n^2$ ($= 4 \times 121 \times 25 = 121 \times 100$) $= 12,100$. Therefore $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$ will be ($= \frac{9216}{21,316} \times aa$) $= \frac{9216}{21,316} \times 25$, and $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$ will be ($= \frac{12,100}{21,316} \times aa$) $= \frac{12,100}{21,316} \times 25$. Therefore $\frac{9216}{21,316} \times 25$, and $\frac{12,100}{21,316} \times 25$, will be two ſquare numbers that will, together, be equal to the given ſquare number 25. And ſo we ſhall eaſily find them to be. For $\frac{9216}{21,316} \times 25 + \frac{12,100}{21,316} \times 25$, are equal to $\frac{9216 + 12,100}{21,316} \times 25 = \frac{21,316}{21,316} \times 25 = 25$.

If we multiply theſe three numbers, $\frac{9216}{21,316} \times 25$, $\frac{12,100}{21,316} \times 25$, and 25, by 21,316, we ſhall thereby obtain the three following whole numbers, to wit, 9216×25, $12,100 \times 25$, and $21,316 \times 25$, which are, all of them, ſquare numbers, and of which the two former are, together, equal to the latter. And, if we divide theſe three whole numbers by 25, we ſhall thereby obtain the three leſſer whole numbers 9216, 12,100, and 21,316, which are, all of them, ſquare numbers, (to wit, the ſquares of the numbers 96, 110, and 146,) and of which the two former are, together, equal to the latter.

Art. 18. We have now obtained, by means of the three general expreſſions $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$, $\frac{4m^2n^2}{\overline{mm + nn}|^2}$, and aa, the twelve following ſets of three whole numbers each, that are, all of them, ſquare numbers, and of which the two firſt numbers in every ſet are, together, equal to the third number; to wit, 1ſt, The numbers 9, 16, and 25;

2dly, The numbers 16, 9, and 25, which differ from the three former numbers 9, 16, and 25, only in the order in which the two firſt numbers 9 and 16 are placed;

3dly, The numbers 225, 64, and 289;

4thly, The numbers 144, 25, and 169;

5thly, The numbers 25, 144, and 169, which differ from the three foregoing numbers only in the order in which the two firſt numbers 25 and 144 are placed;

6thly, The numbers 441, 400, and 841;

7thly, The numbers 2025, 784, and 2809;

8thly, The numbers 256, 900, and 1156;

9thly, The numbers 1600, 1764, and 3364;

10thly, The numbers 12,544, 4356, and 16,900;

11thly, The numbers 576, 4900, and 5476; and,

12thly, The numbers 9216, 12,100 and 21,316.

Art. 19. The ſquare-roots of the foregoing twelve ſets of numbers are as follows; to wit,

1ſt, The numbers 3, 4, and 5;

2dly, The numbers 4, 3, and 5;

3dly, The numbers 15, 8, and 17;

4thly, The numbers 12, 5, and 13;

5thly, The numbers 5, 12, and 13;

6thly, The numbers 21, 20, and 29;

7thly, The numbers 45, 28, and 53;

8thly, The numbers 16, 30, and 34;

9thly, The numbers 40, 42, and 58;

10thly, The numbers 112, 66, and 130;

11thly, The numbers 24, 70, and 74; and

12thly, The numbers 96, 110, and 146.

Art. 20. If we divide the numbers of ſome of the foregoing

going ſets of numbers by 2, (which will not alter the proportion of ſuch numbers to each other,) the ſaid twelve ſets of numbers will be as follows; to wit,

1ſt,	3,	4,	and	5;
2dly,	4,	3,	and	5;
3dly,	15,	8,	and	17;
4thly,	12,	5,	and	13;
5thly,	5,	12,	and	13;
6thly,	21,	20,	and	29;
7thly,	45,	28,	and	53;
8thly,	8,	15,	and	17;
9thly,	20,	21,	and	29;
10thly,	56,	33,	and	65;
11thly,	12,	35,	and	37; and
12thly,	48,	55,	and	73.

And all theſe twelve ſets of numbers will expreſs the lengths of the ſides of different right-angled triangles.

Art. 21. In the three foregoing general expreſſions $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$, $\frac{4m^2n^3}{\overline{mm + nn}|^2} \times aa$, and aa, or $\frac{\overline{mm + nn}|^2}{\overline{mm + nn}|^2} \times aa$, obtained in art. 12, the letter n anſwers to the letter e in the three former general expreſſions $nn - ee$, $2ne$, and $nn + ee$, obtained in art. 4; and the letter m in the three expreſſions obtained in art. 12, anſwers to the letter n in the three former expreſſions obtained in art. 4. And accordingly we find that, where the ſame two numbers have been ſubſtituted inſtead of the letters m and n in the general expreſſions $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$, $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$, and $\frac{\overline{mm + nn}|^2}{\overline{mm + nn}|^2} \times aa$, as were ſubſtituted in art. 4, inſtead of the letters n and e reſpectively, in the general expreſſions $nn - ee$, $2ne$, and $nn + ee$, they have produced the ſame three numbers to repreſent the lengths of the three ſides of a right-angled triangle.

triangle. The only difference between these two sets of general expressions is, that the three expressions $\frac{\overline{mm - nn}|^2}{\overline{mm + nn}|^2} \times aa$, $\frac{4m^2n^2}{\overline{mm + nn}|^2} \times aa$, and $\frac{\overline{mm + nn}|^2}{\overline{mm + nn}|^2} \times aa$, give us the squares of such numbers as will express the sides of right-angled triangles, and the three expressions $nn - ee$, $2ne$, and $nn + ee$, give us the said numbers themselves.

Art. 22. The whole numbers that express the lengths of the sides of a right-angled triangle, cannot, when they are reduced to the lowest numbers possible by dividing them by their common divisors, be, all of them, even numbers.

For, if they were all even numbers, they might all be divided by 2, either once, or more than once, till at last some of the quotients would be odd numbers. Thus, for example, the three even numbers 16, 30, and 34, which have been found above to express the three sides of a right-angled triangle, are all divisible by 2, and are by such division reduced to the three lesser numbers 8, 15, and 17, of which the two latter are odd numbers.

Art. 23. And further, the said numbers that express the lengths of the sides of a right-angled triangle, cannot be, all of them, odd numbers.

For, if the two numbers expressing the lengths of the two sides of the triangle that contain the right angle, were, both of them, odd numbers, their squares would also be odd numbers; because the square of every odd number is an odd number taken an odd number of times, and consequently must be an odd number: and consequently the sum of the said two squares must be an even number, because two odd numbers added together always make an even number. Therefore the square of the number representing the hypotenuse of the triangle, being equal to the said sum of the two other squares, must be an even number. And

And consequently the square-root of the said number, that is, the number representing the hypotenuse of the triangle, must be an even number likewise, which is contrary to the supposition. Therefore it is impossible that all the three numbers which represent the lengths of the sides of a right-angled triangle, should be odd numbers.

Art. 24. There is also another way of finding several whole numbers that shall represent the lengths of the sides of different right-angled triangles; which consists in forming a list, or table, of the squares of the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, &c, set down in their proper order; and a list of the differences of the said squares, (which, it is well known, are the several odd numbers 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, &c, taken in their natural order,) and adding together the said differences that follow any given square number in the list of squares till their sum amounts to another square number. Such a table, carried as far as the square of 100, will be as follows.

A TABLE

A TABLE of the Squares of the Natural Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, *&c; as far as* 100, *and of their ſeveral Differences from each other, and likewiſe of the Differences of thoſe Differences, or of the Second Differences of the ſaid Squares.*

The Natural Numbers.	The Squares of the Natural Numbers.	The Differences of the ſaid Squares.	Their 2d Differences, or the Differences of their Differences.
1	1		
2	4	3	2
3	9	5	2
4	16	7	2
5	25	9	2
6	36	11	2
7	49	13	2
8	64	15	2
9	81	17	2
10	100	19	2
11	121	21	2
12	144	23	2
13	169	25	2
14	196	27	2
15	225	29	2
16	256	31	2
17	289	33	2
18	324	35	2
19	361	37	2
20	400	39	2
21	441	41	2
22	484	43	2
23	529	45	2
24	576	47	2
25	625	49	2
26	676	51	2
27	729	53	2
28	784	55	2
		57	2

The Natural Numbers.	The Squares of the Natural Numbers.	The Differences of the said Squares.	Their 2d Differences, or the Differences of their Differences.
29	841		
30	900	59	2
31	961	61	2
32	1024	63	2
33	1089	65	2
34	1156	67	2
35	1225	69	2
36	1296	71	2
37	13 9	73	2
38	1444	75	2
39	1521	77	2
40	1600	79	2
41	1681	81	2
42	1764	83	2
43	1841	85	2
44	1936	87	2
45	2025	89	2
46	2116	91	2
47	2209	93	2
48	2304	95	2
49	2401	97	2
50	2500	99	2
51	2601	101	2
52	2704	103	2
53	2809	105	2
54	2916	107	2
55	3025	109	2
56	3136	111	2
57	3249	113	2
58	3364	115	2
59	3481	117	2
60	3600	119	2
61	3721	121	2
62	3844	123	2
63	3969	125	2
64	4096	127	2
		129	2

The

The Natural Numbers.	The Squares of the Natural Numbers.	The Differences of the said Squares.	Their 2d Differences, or the Differences of their Differences.
65	4225	131	
66	4356	133	2
67	4489	135	2
68	4624	137	2
69	4761	139	2
70	4900	141	2
71	5041	143	2
72	5184	145	2
73	5329	147	2
74	5476	149	2
75	5625	151	2
76	5776	153	2
77	5929	155	2
78	6084	157	2
79	6241	159	2
80	6400	161	2
81	6561	163	2
82	6724	165	2
83	6889	167	2
84	7056	169	2
85	7225	171	2
86	7396	173	2
87	7569	175	2
88	7744	177	2
89	7921	179	2
90	8100	181	2
91	8281	183	2
92	8464	185	2
93	8649	187	2
94	8836	189	2
95	9025	191	2
96	9216	193	2
97	9409	195	2
98	9604	197	2
99	9801	199	2
100	10,000		

Art. 25. In the foregoing table the firſt column contains all the natural numbers 1, 2, 3, 4, 5, &c, as far as 100; and the ſecond column contains the ſquares of the ſaid numbers ſet down even with the ſaid numbers themſelves, or ſo that every ſquare number ſhall be in the ſame horizontal line with the natural number of which it is the ſquare; and the third column contains the differences of the ſquare numbers in the ſecond column, ſet down in lines between the lines in which the ſquares themſelves, of which they are the differences, are ſet down; and the fourth column contains the differences of the foregoing differences that are ſet down in the third column, or the ſecond differences of the ſquare numbers that are ſet down in the ſecond column; and each of theſe ſecond differences is ſet down in a line that lies between the two lines in which the two firſt differences, of which it is the difference, are ſet down. And we may obſerve, that the differences ſet down in the third column are the odd numbers 3, 5, 7, 9, 11, 13, 15, &c, taken in their natural order; and the ſecond differences ſet down in the fourth column, being the differences of the ſaid odd numbers, are all equal to each other, and to the number 2.

Art. 26. From the conſtruction of this table, it is obvious that, if we take any number whatſoever in the firſt column, and look out its ſquare in the ſecond column, and then add together the ſeveral differences in the third column that follow the ſaid ſquare number in the ſecond column, till the ſum of the ſaid differences ſhall amount to a ſquare number, the ſquare-root of the ſaid ſquare number, or the number in the firſt column that is placed even with it, and the number firſt taken, will expreſs the lengths of the two ſides of a right-angled triangle that contain the right angle, and the number in the firſt column that immediately follows the laſt of the ſaid differences in the third column, ſo added together, will expreſs the length of the hypotenuſe of the ſame triangle. Thus, for example, if we take the number 3 in the firſt column, and find its ſquare, to wit, 9, in the ſecond column, and then add up the

the differences 7, 9, 11, 13, &c, in the third column, which come after the ſquare number 9, till their ſum becomes equal to a ſquare number, (for which purpoſe we need only add together the two differences 7 and 9, becauſe their ſum is 16, which is a ſquare number,) the ſquare root of the ſaid ſquare number 16, or the number in the firſt column which is placed even with it, to wit, the number 4, and the number 3, which was taken at firſt, will expreſs the lengths of the two ſides of a right-angled triangle that contain the right angle, and the number 5 in the firſt column, which immediately follows the laſt of the ſaid two differences, ſo added together, to wit, 9, will expreſs the length of the hypotenuſe of the ſame triangle. So that we we ſhall hereby obtain the whole numbers 3, 4, and 5, to expreſs the lengths of the three ſides of a right-angled triangle: which numbers we had before obtained by both the former methods of inveſtigation.

Art. 27. If we look in the 3d column, or column of differences, for thoſe differences that are themſelves ſquare numbers, without being added to any of the foregoing, or following, differences to make them ſo, (which differences are but few in number, being only the following ſix numbers, to wit, 9, 25, 49, 81, 121, and 169, in the whole table,) we may at once obtain from each of theſe differences a ſet of numbers that will expreſs the lengths of the three ſides of a right-angled triangle.

For, ſince 9 is a ſquare number, and is likewiſe the difference between the two contiguous ſquare numbers 16 and 25, and conſequently 9 + 16 are = 25, it follows that the ſquare-roots of theſe three numbers 9, 16, and 25, that is, the three numbers 3, 4, and 5, will expreſs the three ſides of a right-angled triangle.

And, in like manner, ſince 25 is a ſquare number, and is likewiſe the difference between the two contiguous ſquare numbers 144 and 169, and conſequently 25 + 144 are = 169, it follows that the ſquare-roots of the three numbers

25,

25, 144, and 169, that is, the three numbers 5, 12, and 13, will expreſs the three ſides of a right-angled triangle.

And, ſince 49 is a ſquare number, and is likewiſe the difference of the two contiguous ſquare numbers 576 and 625, and conſequently 49 + 576 are equal to 625, it follows that the ſquare-roots of the three numbers 49, 576, and 625, that is, the three numbers 7, 24, and 25, will expreſs the three ſides of a right-angled triangle.

And, ſince 81 is a ſquare number, and is likewiſe the difference of the two contiguous ſquare numbers 1600 and 1681, and conſequently 81 + 1600 are = 1681, it follows that the ſquare-roots of the three numbers 81, 1600, and 1681, that is, the three numbers 9, 40, and 41, will expreſs the three ſides of a right-angled triangle.

And, ſince 121 is a ſquare number, and is likewiſe the difference of the two contiguous ſquare numbers 3600 and 3721, and conſequently 121 + 3600 are = 3721, it follows that the ſquare-roots of the three numbers 121, 3600, and 3721, that is, the three numbers 11, 60, and 61, will expreſs the three ſides of a right-angled triangle.

And, laſtly, ſince 169 is a ſquare number, and likewiſe is the difference of the two contiguous ſquare numbers 7056 and 7225, and conſequently 169 + 7056 are = 7225, it follows that the ſquare-roots of the three numbers 169, 7056, and 7225, that is, the three numbers 13, 84, and 85, will expreſs the three ſides of a right-angled triangle.

Art. 28. In this way of obtaining three numbers that ſhall expreſs the three ſides of a right-angled triangle, namely, by chuſing ſuch numbers in the 3d column, or column of differences, as are themſelves ſquare numbers, it is evident that the number expreſſing the hypotenuſe of the triangle will always exceed the greater of the other two numbers, that expreſs its ſides, by an unit. But, when we take ſeveral ſucceſſive differences, of which the ſum is equal to a ſquare number, the number that expreſſes the hypotenuſe of the triangle, will exceed the number taken at firſt, and between

between the ſquare of which, and the ſquare of the number repreſenting the hypotenuſe, the ſeveral differences that are added together lie, by as many units as there are differences that have been ſo added together in order to make a ſquare number. Of this it will not be amiſs to give a few examples.

Art. 29. Let us take 8 for the firſt number. Then, ſince the ſquare of 8 is 64, we muſt begin with the difference 17, which comes immediately after 64, and we muſt add together this difference 17, and the following differences 19, 21, 23, 25, 27, &c, till their ſum amounts to a ſquare number. For this purpoſe we need add together only two of theſe differences, to wit, 17 and 19. For 17 + 19 are = 36, which is a ſquare number, to wit, the ſquare of 6. Therefore the firſt number 8, and the number 6, (or the ſquare-root of the ſum of theſe two differences), and the number 10, (or the ſquare-root of the ſquare number 100, which comes immediately after the laſt difference 19), will be three numbers that will expreſs the three ſides of a right-angled triangle. For $\overline{8}|^2 + \overline{6}|^2$ will be $= \overline{10}|^2$, or 64 + 36 will be = 100. And the number 10, (which repreſents the hypotenuſe of the triangle,) exceeds the firſt number 8, (which repreſents the greater of its two ſides,) by 2, or two units, or the ſame number of units as there were differences added together, in order to produce the ſquare number 36.

And, if, inſtead of taking only two of the differences 17, 19, 21, 23, 25, 27, 29, &c, we take nine of them, we ſhall find their ſum to be equal to another ſquare number, to wit, 225, which is the ſquare of 15. For 17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 + 33 are = 225. Therefore the firſt number 8, and the number 15, (or the ſquare-root of the ſum of theſe nine differences,) and the number 17, (or the ſquare-root of the ſquare number 289, which comes immediately after the laſt difference 33,) will be three numbers that will expreſs the three ſides of a right-angled triangle. For $\overline{8}|^2 + \overline{15}|^2$ will be $= \overline{17}|^2$, or 64 +

225 will be = 289. And the number 17, (which repreſents the hypotenuſe of the triangle,) exceeds the firſt number 8, (which repreſents one of the ſides that incloſe the right angle,) by 9, or nine units, or the ſame number of units as there were differences added together, in order to produce the ſquare number 225.

Art. 30. Now let the firſt number be 20. Then, ſince the ſquare of 20 is 400, we muſt begin with the difference 41, which comes immediately after 400, and muſt add together this difference 41, and the following differences 43, 45, 47, 49, 51, &c, till their ſum amounts to a ſquare number. And for this purpoſe it will be neceſſary to add together nine of theſe differences. For 41 + 43 + 45 + 47 + 49 + 51 + 53 + 55 + 57 are = 441, which is the ſquare of 21. Therefore the firſt number 20, and the number 21, (or the ſquare-root of the ſum of theſe nine differences,) and the number 29, (or the ſquare-root of the ſquare number 841, which comes immediately after the laſt difference 57,) will be three numbers that will expreſs the three ſides of a right-angled triangle. For $\overline{20}|^2 + \overline{21}|^2$ will be $= \overline{29}|^2$, or 400 + 441 will be = 841. And the number 29, (which repreſents the hypotenuſe of the triangle,) exceeds the firſt number 20, (which repreſents one of the ſides that incloſe the right-angle,) by 9, or nine units, or the ſame number of units as there were differences added together in order to produce the ſquare number 441.

Art. 31. Now let the firſt number be 28. Then, ſince the ſquare of 28 is 784, we muſt begin with the difference 57, which comes immediately after 784, and we muſt add together this difference 57, and the following differences 59, 61, 63, 65, &c, till their ſum amounts to a ſquare number. And for this purpoſe it will be neceſſary to add together ſeven of theſe differences. For 57 + 59 + 61 + 63 + 65 + 67 + 69 are = 441, which is the ſquare of 21. Therefore the firſt number 28, and the number 21, (or (or the ſquare-root of the ſum of theſe ſeven differences,) and the number 35, (or the ſquare-root of the ſquare number 1225, which comes immediately after the laſt difference 69,) will be three numbers that will expreſs the three ſides of a right-angled triangle. For $\overline{28}|^2 + \overline{21}|^2$ will be $= \overline{35}|^2$, or $784 + 441$ will be $= 1225$. And the number 35, (which repreſents the hypotenuſe of the triangle,) exceeds the firſt number 28, (which repreſents one of the ſides that incloſe the right angle,) by 7, or ſeven units, or the ſame number of units as there were differences added together, in order to produce the ſquare number 441.

Theſe examples, I apprehend, are ſufficient to explain this method of obtaining different ſets of whole numbers that ſhall expreſs the lengths of the ſides of different right-angled triangles. And with them I ſhall conclude this little tract.

End of the Diſcourſe concerning the Methods of finding Rational Numbers that expreſs the Sides of Right-angled Triangles.

OF THE

DIFFERENCES

OF THE

CUBES

OF THE

NATURAL NUMBERS 1, 2, 3, 4, 5, 6, 7, &c.

Article 1. We have ſeen in the table of the ſquares of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, ſet down in the foregoing Tract, that the firſt differences of the ſquares of thoſe numbers are the odd numbers 3, 5, 7, 9, 11, 13, 15, 17, &c, in their natural order, and their ſecond differences, or the differences of their firſt differences, are all equal to each other, and to the number 2. We will now ſet down a table of the cubes of the ſeveral natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, and of their differences, and the differences of thoſe firſt differences, and the differences of thoſe ſecond differences; by which it will appear that the cubes of the ſaid numbers have three orders of differences, and that their differences of the third order are all equal to each other and to the number 6. This Table will be as follows:

A TABLE of the Cubes of the Natural Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, *&c, as far as* 100; *together with their First, Second, and Third, Differences.*

The Natural Numbers	Their Cubes.	The Differences of their Cubes.	Their 2d Diffs. or the Diffs. of the former Diffs.	The 3d Differences of the said Cubes.
1	1	7	12	6
2	8	19	28	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	6
6	216	127	42	6
7	343	169	48	6
8	512	217	54	6
9	729	271	60	6
10	1000	331	66	6
11	1331	397	72	6
12	1728	469	78	6
13	2197	547	84	6
14	2744	631	90	6
15	3375	721	96	6
16	4096	817	102	6
17	4913	919	108	6
18	5832	1027	114	6
19	6859	1141	120	6
20	8000	1261	126	6
21	9261	1387	132	6
22	10,648	1519	138	6
23	12,167	1657	144	6
24	13,824	1801	150	6
25	15,625	1951	156	6
26	17,576	2107	162	6
27	19,683	2269	168	6
28	21,952	2437	174	6
29	24,389	2611	180	

The

The Natural Numbers	Their Cubes.	The Differences of their Cubes.	Their 2d Diffs. or the Diffs. of the former Diffs.	The 3d Differences of the said Cubes.
30	27,000	2791	186	
31	29,791	2977	192	6
32	32,768	3169	198	6
33	35,937	3367	204	6
34	39,304	3571	210	6
35	42,875	3781	216	6
36	46,656	3997	222	6
37	50,653	4219	228	6
38	54,872	4447	234	6
39	59,319	4681	240	6
40	64,000	4921	246	6
41	68,921	5167	252	6
42	74,088	5419	258	6
43	79,507	5677	264	6
44	85,184	5941	270	6
45	91,125	6211	276	6
46	97,336	6487	282	6
47	103,823	6769	288	6
48	110,592	7057	294	6
49	117,649	7351	300	6
50	125,000	7651	306	6
51	132,651	7957	312	6
52	140,608	8269	318	6
53	148,877	8587	324	6
54	157,464	8911	330	6
55	166,375	9241	336	6
56	175,616	9577	342	6
57	185,193	9919	348	6
58	195,112	10267	354	6
59	205,379	10621	360	6
60	216,000	10981	366	6
61	226,981	11347	372	6
62	238,328	11719	378	6
63	250,047	12097	384	6
64	262,144	12481	390	6

The

The Natural Numbers	Their Cubes.	The Differences of their Cubes.	Their 2d Diffs. or the Diffs. of the former Diffs.	The 3d Differences of the said Cubes.
65	274,625	12871	396	6
66	287,496	13267	402	6
67	300,763	13669	408	6
68	314,432	14077	414	6
69	328,509	14491	420	6
70	343,000	14911	426	6
71	357,911	15337	432	6
72	373,248	15769	438	6
73	389,017	16207	444	6
74	405,224	16651	450	6
75	421,875	17101	456	6
76	438,976	17557	462	6
77	456,533	18019	468	6
78	474,552	18487	474	6
79	493,039	18961	480	6
80	512,000	19441	486	6
81	531,441	19927	492	6
82	551,368	20419	498	6
83	571,787	20917	504	6
84	592,704	21421	510	6
85	614,125	21931	516	9
86	636,056	22447	522	6
87	658,503	22969	528	6
88	681,472	23497	534	6
89	704,969	24031	540	6
90	729,000	24571	546	6
91	753,571	25117	552	6
92	778,688	25669	558	6
93	804,357	26227	564	6
94	830,584	26791	570	6
95	857,375	27361	576	6
96	884,736	27937	582	6
97	912,673	28519	588	6
98	941,192	29107	594	
99	970,299	29701		
100	1,000,000			

Art. 2.

Art. 2. And in like manner it will be found that the fourth powers of the natural numbers 1, 2, 3, 4, 5, &c, will have four orders of differences, and that their fifth powers will have five orders of differences; and, in general, that their *n*th powers, *n* being any whole number whatſoever, will have *n* orders of differences. This is a curious property of the powers of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, and has been long known to Mathematicians. The celebrated Mr. Leibnitz, of Hanover, had taken notice of it before the month of February, 1673; and it had been obſerved before him by a French Mathematician, named *Mouton*, (who was a Canon of the Church of Lyons,) in a book on the apparent diameters of the Sun and Moon; but which Mr. Leibnitz declared he had not ſeen at the time he made the ſame diſcovery. Mr. Leibnitz's manner of conſidering the ſubject is explained pretty fully in a large extract from a letter of his to Mr. Henry Oldenburgh, the Secretary of the Royal Society of London, dated at London on the 3d of February, $167\frac{2}{3}$, which has been publiſhed in the *Commercium Epiſtolicum* of Mr. John Collins and other Mathematicians of that time. This extract, as it contains ſeveral intereſting particulars relating to theſe numbers, I ſhall here inſert at length from the ſaid *Commercium Epiſtolicum*, pages 108, 109, 110, - - - - 114. It is as follows.

Art. 3. *Excerpta ex Epiſtolâ Domini* Gothofredi Gulielmi Leibnitzii *ad Dominum* Oldenburgh, *Londini, Anno* $167\frac{2}{3}$, 3^tio^ *die Februarii, datâ.*

Hujus Autographon in ſcriniis Regiæ Societatis extat, et exemplar ejus in Libro Epiſtolarum dictæ Societatis, N°. 6, pag. 35, deſcriptum legitur.

CUM heri apud illuſtriſſimum *Boylium* incidiſſem in clariſſimum *Pellium* Mathematicum inſignem, ac de Numeris incidiſſet

incidiſſet mentio, commemoravi ego, ductus occaſione Sermonum, eſſe mihi methodum ex quodam differentiarum genere, quas voco *generatrices*, colligendi terminos Seriei cujuſcunque continuè creſcentis vel decreſcentis. Differentias autem generatrices voco, ſi datæ Seriei inveniantur differentiæ, & differentiæ differentiarum, & ipſarum ex differentiis differentiarum differentiæ, *&c.* & ſeries conſtituatur ex termino primo & primâ differentiâ, & primâ differentiâ differentiarum, & primâ differentiâ ex differentiis differentiarum, *&c.* ea Series erit differentiarum generatricium, ut ſi Series continuè creſcens vel decreſcens fuerit a, b, c, d.

Poſitâ $\backsim$ *differentiæ Notâ,*] differentiæ generatrices erunt:

$1\,a \;.\; 2\,a \backsim b \;.\; 3\,\overline{a \backsim b} \backsim \overline{b \backsim c} \;.\; 4\,\overline{\overline{a \backsim b} \backsim \overline{b \backsim b}} \backsim \overline{\overline{b \backsim c} \backsim \overline{c \backsim d}}.$

$$
\begin{array}{lccccccc}
4 & & & \overline{\overline{a \backsim b} \backsim \overline{b \backsim c}} \backsim \overline{\overline{b \backsim c} \backsim \overline{c \backsim d}} & & & & \\
3 & & & a \backsim b \backsim b \backsim c & & b \backsim c \backsim c \backsim d & & \\
2 & & a \backsim b & & b \backsim c & & c \backsim d & \\
1 & a & & b & & c & & d
\end{array}
$$

Aut in Numeris; ſi Series ſit Numerorum cubicorum deinceps ab unitate creſcentium, differentiæ generatrices erunt numeri 0, 1, 6, 6. Voco autem *generatrices*, quia ex iis certo modo multiplicatis producuntur termini Seriei; cujus uſus tum maximè apparet, cum differentiæ generatrices ſunt finitæ, termini autem Seriei infiniti; ut in propoſito exemplo Numerorum Cubicorum.

$$
\begin{array}{ccccccccccccc}
 & & & 0 & & 0 & & 0 & & & & & \\
 & & 6 & & 6 & & 6 & & 6 & & & & \\
 & 6 & & 12 & & 18 & & 24 & & 30 & & & \\
1 & & 7 & & 19 & & 37 & & 61 & & 91 & & \\
0 & 1 & & 8 & & 27 & & 64 & & 125 & & 216 &
\end{array}
$$

Hoc cum audiſſet clariſſimus *Pellius*, reſpondit, id jam fuiſſe in literas relatum à D. *Mouton*, Canonico *Lugdunenſi*, ex obſer-

obſervatione nobiliſſimi viri *Franciſci Regnaldi Lugdunenſis*, dudùm in literario Orbe celebris, in libro laudati D. *Mouton* de diametris apparentibus Solis & Lunæ. Ego qui ex Epiſtolâ quâdam à *Regnaldo* ad *Monconiſium* ſcriptâ, & Diario itinerum *Monconiſiano* inſertâ, nomen D. *Moutoni* & deſignata ejus duo didiceram; Diametros Luminarium apparentes, & conſilium de menſuris rerum ad poſteros tranſmittendis; ignorabam tamen librum ipſum prodiiſſe: quarè apud D. *Oldenburgium Societatis Regalis* Secretarium, ſumtum mutuò tumultuariè percurri, & inveni veriſſimè dixiſſe *Pellium*. Sed & mihi tamen dandam operam credidi, ne qua in animis relinqueretur ſuſpicio, quaſi, tacito * inventoris nomine, alienis meditationibus honorem mihi quærere voluiſſem; & ſpero appariturum eſſe, non adeò egenum me meditationum propriarum ut cogar alienas emendicare. Duobus autem argumentis ingenuitatem meam vindicabo. Primò, ſi ipſas Schedas meas confuſas, in quibus non tantùm inventio mea ſed & inveniendi modus occaſióque apparet, monſtrem: deinde, ſi quædam momenti maximi *Regnaldo Moutonóque* indicta addam, quæ ab heſterno veſpere confinxiſſe me non ſit veriſimile, quaéque non poſſunt facilè expectari à Tranſcriptore.

Ex Schedis meis occaſio inventi hæc apparet: quærebam modum inveniendi differentias omnis generis poteſtatum, quemadmodum conſtat differentias Quadratorum eſſe numeros impares; inveneràmque regulam generalem ejuſmodi.

Datâ potentiâ gradûs dati præcedente, invenire ſequentem (vel contrà) diſtantiæ datæ vel radicum datarum; ſeu invenire potentiarum gradûs dati utcunque diſtantium differentias. Multiplicetur potentia gradûs, proximè præcedentis radicis majoris per differentiam radicum; & differentia potentiarum gradûs proximè præcedentis multiplicetur per radicem minorem: productorum ſumma erit quæſita differentia potentiarum, quarum radices ſunt datæ. Eandem regulam ita inflexeram, ut ſufficeret, præter radices, cujuſlibet gradûs, etiamſi non proximè præcedentis, potentias datarum radicum dari, ad differentias potentiarum alterius cu-

* Id eſt, celato.

juſcunque, licet altioris, gradûs inveniendas. Et oſtendi quod in Quadratis obſervatur, numeros impares eſſe eorum differentias, id non niſi regulæ propoſitæ ſubſumptionem eſſe.

His meditationibus defixus, quemadmodum in Quadratis differentiæ ſunt numeri impares, ita quoque quæſivi quales eſſent differentiæ Cuborum, quæ cum irregulares viderentur, quæſivi differentias differentiarum, donec inveni differentias tertias eſſe numeros ſenarios. Hæc obſervatio mihi aliam peperit: videbam enim ex differentiis præcedentibus generari terminos differentiáſque ſequentes, ac proinde, ex primis, quas ideò voco *generatrices*, (ut hoc loco o . 1 . 6 . 6,) ſequentes omnes. Hoc concluſo, reſtabat invenire, quo additionis, multiplicationíſve, aut horum complicationis, genere, termini ſequentes ex differentiis generatricibus producerentur. Atque ita reſolvendo experiundóque deprehendi primum Terminum o componi ex primâ differentiâ generatrice o ſumtâ ſemèl, ſeu vice unâ: Secundum 1 ex primâ o ſemèl & ſecundâ 1 ſemèl: Tertium 8 ex primâ o ſemèl, ſecundâ 1 bis & tertiâ 6 ſemèl: nam $0 \times 1 + 1 \times 2 + 6 \times 1 = 8$. Quartum 27, ex primâ o ſemèl, ſecundâ 1 tèr, tertiâ 6 tèr, quartâ 6 ſemèl: nam $0 \times 1 + 1 \times 3 + 6 \times 3 + 6 \times 1 = 27$, *&c.* ídque Analyſis mihi univerſale eſſe comprobavit. Hæc fuit occaſio obſervationis meæ, longè alia à *Moutonianâ*, qui cum in Tabulis condendis laboraret, in hoc calculandi compendium cum *Regnaldo* incidit: nec vel illi vel *Regnaldo* adimenda laus; quod & *Briggius* in Logarithmicis ſuis jam olim talia quædam, obſervante *Pellio*, ex parte advertit. Mihi hoc ſupereſt ut addam nonnulla illis indicta, ad amoliendum Tranſcriptoris nomen; neque enim intereſt Reipublicæ quis obſervaverit, intereſt quid obſervetur. Primum, ergò illud adjicio, quod apud *Moutonium* non extat, & caput tamen rei eſt: quinam ſint illi numeri, quorum Tabulam ille exhibet in infinitum continuandam, quorum ductu in differentias generatrices, productis inter ſe junctis, termini Serierum generentur. Vides enim ex ipſo modo quo tabula ab eo *pag.* 385, exhibetur, non fuiſſe id ei ſatìs exploratum; alioqui enim veriſimile eſt ita Tabulam fuiſſe diſpoſiturum, ut ea numerorum connexio

atque

atque harmonia appareret; niſi quis de induſtriâ texiſſe dicat: ita enim ſe habet pars Tabulæ.

1	1					
2	1	1				
3	1	2	1			
(4)	1	3	3	1		
5	1	4	6	4	1	
6	1	5	10	10	5	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

Apparet ex hujus Tabulæ conſtructione ſolam haberi rationem correſponſûs numerorum generantium cum numero Termini generati; ut cum terminus eſt quartus (4) producitur ex primâ differentiâ ſemèl, ſecundâ ter 3, tertiâ ter 3, quartâ ſemèl 1; ideò in eâdem (4) Lineâ tranſversâ locantur 1. 3. 3. 1. Sed vel non obſervavit vel diſſimulavit autor correſponſum numerorum, ſi à ſummo deorſùm eundo per columnas diſponantur hoc modo,

1	1					
2	1 ↘	1				
3	1	2 ↘	1			
4	1	3	3 ↘	1		
5	1	4	6	4 ↘	1	
6	1	5	10	10	5 ↘	1
7	1	6	15	20	15	6
8	1	7	21	35	35	21
9	1	8	28	56	70	56
10	1	9	36	84	126	126
11	1	10	45	120	210	252

Ita enim ſtatìm vera genuináque eorum natura ac generatio apparet; eſſe, ſcilicet, eos numeros quos *Combinatorios* appellare ſoleo, de quibus multa dixi in diſſertatiunculâ de Arte Combinatoriâ; quóſque alii appellant Ordines numericos; alii ſpecie primam columnam Unitatum; ſecundam Numerorum naturalium, tertiam Triangularium, quartam Pyramidalium, quintam Triangulo-Triangularium, *&c.* de quibus integer extat Tractatus *Paſchalii* ſub titulo Trianguli Arithmetici; in quo tamen proprietatem numerorum ejuſmodi tàm illuſtrem támque naturalem * non obſervatam ſum miratus. Sed eſt profectò caſus quidam in inveniendo, qui non ſemper maximis ingeniis maxima, ſed ſæpe etiam mediocribus nonnulla offert.

Hinc jam vera numerorum iſtorum natura, & Tabulæ conſtructio, ſive à *Regnaldo* ſive à *Moutonio* diſſimulata, intelligitur: ſemper enim terminus datus columnæ datæ componitur ex termino præcedente columnæ tàm præcedentis quàm datæ: Atque illud quoque apparet, non opus eſſe moleſto calculo ad Tabulam à *Moutonio* propoſitam continuandam, ut ipſe poſtulat; cum hæ numerorum Series paſsìm jam tradantur calculentúrque.

Cæterùm *Moutonius* obſervatione iſtâ ad interponendas medias proportionales inter duos extremos numeros datos; ego ad inveniendos ipſos numeros extremos in infinitum cum eorum differentiis, utendum cenſebam. Hinc ille, non niſi cum differentiæ ultimæ evaneſcunt (aut penè evaneſcunt) uſum regulæ invenit; ego detexi innumerabiles caſus, regulâ quâdam inobſervatâ comprehendendos; ubi poſſum ex datis numeris finitis certo modo multiplicatis producere numeros plurimarum Serierum in infinitum euntium, etſi differentiæ earum non evaneſcant.

Ex iiſdem fundamentis poſſum efficere in progreſſionibus problemata plurima; aut in Numeris ſingularibus, aut in Rationibus vel Fractionibus: poſſum enim progreſſiones ad-

* *Imò obſervata fuit. Vide* Paſchalii *Triangulum Arithmeticum*, Pariſiis *Anno* 1665 *editum, pag.* 2. *ubi definitionum antepenultima hæc eſt.*

Le nombre de chaque cellule eſt egal à celuy de la cellule qui la précéde dans ſon rang perpendiculaire, plus à celuy de la cellule qui la précéde dans ſon rang parallèle. Ainſi la cellule F, *c'eſt à dire le nombre de la cellule* F, *égale la cellule* C *plus la cellule* E; *&* *ainſi des autres.*

dere

dere ſubtraheréque, imò multiplicare quoque & dividere, ídque compendiosè.

$\frac{1}{3}$.	$\frac{1}{4}$.	$\frac{1}{5}$.	$\frac{1}{6}$
$\frac{1}{6}$.	$\frac{1}{10}$.	$\frac{1}{15}$.	$\frac{1}{21}$
$\frac{1}{10}$.	$\frac{1}{20}$.	$\frac{1}{35}$.	$\frac{1}{56}$
$\frac{1}{15}$.	$\frac{1}{35}$.	$\frac{1}{70}$.	$\frac{1}{126}$
&c.	*&c.*	*&c.*	*&c.*

Multa alia circà hos numeros obſervata ſunt à me, ex quibus illud eminet, quod modum habeo ſummam inveniendi Seriei Fractionum in infinitum decreſcentium; quarum numerator Unitas, nominatores vero numeri iſti Triangulares aut Pyramidales, aut Triangulo-Triangulares, *&c.*

End of the Extract from Mr. Leibnitz'*s Letter.*

Art. 4. By the help of the foregoing table of the cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, as far as 100, we may find the cube-root of any number exact to two places of figures, without the trouble of any calculation whatſoever, or by the mere inſpection of the table. Thus, for example, if I wanted to find the cube-root of 2000, I need only look along the ſecond column of the table, (which contains the cubes of the numbers ſet down in the firſt column,) till I found the two cube numbers which are neareſt to the propoſed number 2000, the one above it and the other below it. Theſe numbers I ſhould find to be 2197 and 1728; of which the former is the cube of 13, and the latter is the cube of 12. And hence I might conclude with certainty that, ſince the propoſed number 2000 is greater than 1728, or the cube of 12, but leſs than 2197, or the cube of 13, its cube-root muſt be greater than 12, but leſs than 13, and conſequently that the two firſt figures of it muſt be 12. And from the two firſt figures of the cube-root of any propoſed number, we may derive the following figures of it to five, or ſix, or any greater number

number of figures that we may defire, by the method of approximation invented for this purpofe by Monfieur *De Lagney*, which has fince been approved and adopted by Dr. *Halley* and other Mathematicians, as the moft convenient that can be taken. This method is as follows.

Monfieur De Lagney'*s Method of approximating to the Value of the Cube-root of any propofed Number, when the Two, or Three, firft Figures of the faid Cube-root are known.*

Art. 5. If the number of which the cube-root is to be extracted be called c, and a number, confifting of two, or more figures, that is fomewhat lefs than the true value of the faid cube-root be called a, the remaining part of the faid cube-root will be very nearly equal to the quantity $\frac{\overline{c-a^3}\times a}{c+2a^3}$, and confequently the whole of the faid cube-root will be very nearly equal to $a+\frac{\overline{c-a^3}\times a}{c+2a^3}$; but it will always be a little greater than the faid quantity. Alfo the faid remaining part of the cube-root of c, which is to be added to its firft value a, will be very nearly equal to the quantity $\sqrt{\frac{4c-a^3}{12a}}-\frac{a}{2}$, and confequently the whole of the faid cube-root will be very nearly equal to $a+\sqrt{\frac{4c-a^3}{12a}}-\frac{a}{2}$, or to $\frac{a}{2}+\sqrt{\frac{4c-a^3}{12a}}$; but it will always be a little lefs than that quantity. And this latter expreffion will be a little nearer to the true value of the cube-root of c than the former expreffion $a+\frac{\overline{c-a^3}\times a}{c+2a^3}$; but the difference will be fo fmall as to be hardly worth confidering.

Art. 6.

Art. 6. And, if a, or the firſt near value of the cube-root of the propoſed number c, be a little greater than its true value, the quantity to be ſubtracted from a, in order to make it equal to the ſaid true value, will be very nearly equal to the quantity $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$, and conſequently the ſaid cube-root will be very nearly equal to $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$; but it will be always a little greater than the ſaid quantity.

Alſo the ſaid difference between a and the true value of the cube-root of c, or quantity which is to be ſubtracted from a, in order to make it equal to the ſaid cube-root, will be very nearly equal to the quantity $\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$, and conſequently the ſaid cube-root will be very nearly equal to $a - \overline{\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}}$, or to $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$; but it will always be a little leſs than the ſaid quantity. And this latter expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be a little nearer to the true value of the cube-root of c than the former expreſſion $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$; but the difference will be ſo ſmall as to be hardly worth conſidering.

Art. 7. The number of figures that will be exact in the ſecond near value of the cube-root of the propoſed number c that will be obtained by either of theſe four expreſſions $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, (which laſt expreſſion, we may obſerve, is the very ſame with the ſecond expreſſion,) is uſually triple, or triple wanting one figure, and in the worſt caſes is triple wanting two figures, of the number of figures that are exact in the firſt near value, a.

An

An Example of the Extraction of the Cube-root of a Number, by means of the foregoing Method of Approximation.

Art. 8. Let it be required to extract the cube-root of the number 2; which anfwers to the folution of the Problem, fo much celebrated amongft the Antients, of doubling the cube, or finding the length of the fide of a cube that fhall be double of a given cube.

Here I obferve, in the firft place, that, fince the number 2000 is equal to 1000 × 2, or to 10 × 10 × 10 × 2, the cube-root of 2000 muft be equal to 10 times the cube-root of 2. But it appears from the foregoing table of cube numbers, that the cube-root of 2000 muft be greater than 12, but lefs than 13, and confequently that the two firft figures of it muft be 12. Therefore the cube-root of 2 muft be $(= \frac{12}{10} = \frac{10}{10} + \frac{2}{10}) = 1.2$, or the two firft figures of the faid cube-root muft be 1.2. Here then we have $c = 2$, or 2.000,000, and $a = 1.2$, and confequently $a^3 = 1.728$, and $2a^3 = 3.456$, and $c + 2a^3$ $(= 2 + 3.456) = 5.456$, and $c - a^3$ $(= 2.000 - 1.728) = 0.272$, and $\overline{c - a^3} \times a$ $(= 0.272 \times 1.2)$ 3.264, and $\frac{\overline{c - a^3} \times a}{c + 2a^3}$ $(= \frac{3.264}{5.456}) = 0.059,82$. Therefore $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be $= 1.2 + 0.059,82$, or 1.259,82; or the fecond near value of the cube-root of 2, which is obtained by means of the firft expreffion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ given in art. 5, is 1.259,82.

The number of figures that are exact in this fecond near value, 1.259,82, of the cube-root of 2, is four, to wit, the figures 1.259, that is, three times as many figures, wanting two, as are contained in 1.2, or *a*, the firft near value of the

the ſaid cube-root, the more accurate value of which is 1.259,921,049, &c, which is greater than 1.259,82 by 0.000,1, &c.

The other expreſſion given in art. 5, to wit, $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, may be computed as follows.

Since a is = 1.2, and c is = 2, we ſhall have 12 a (= 12 × 1.2) = 14.4, and $\frac{a}{2}$ = 0.6, and $4c$ = 8.000, and $4c - a^3$ (= 8.000 — 1.728) = 6.272, and $\frac{4c - a^3}{12a}$ (= $\frac{6.272}{14.4}$) = 0.435,555,555, &c, and $\sqrt{\frac{4c - a^3}{12a}}$ (= $\sqrt{}$ 0.435, 555,555, &c,) = 0.659,96, and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ (= 0.6000,00 + 0.659,96) = 1.259,96. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, or the ſecond near value of the cube-root of 2, which is obtained by means of the ſecond expreſſion given in art. 5, is 1.259,96; which is exact in the firſt five figures 1.2599, and is greater than the true value of the ſaid cube-root, to wit, 1.259,92, &c, by only 0.000,04, &c, which is leſs than the difference 0.000,1, by which 1.259,82, or the ſecond value of the cube-root of 2, obtained by means of the former expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, fell ſhort of the true value of the ſaid cube root. But either of theſe ſecond near values of the ſaid cube-root, 1.259,82 and 1.259,96, is a great improvement upon its firſt near value, 1.2, though leſs than it would have been if the two firſt figures of the cube-root of the propoſed number had been higher figures than 1 and 2.

Art. 9. And, if we repeat this proceſs of approximation with either of the two expreſſions $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, and $\frac{a}{2}$ +

$\sqrt{\frac{4c-a^3}{12a}}$, taking the firſt four figures of the ſecond values of the cube-root of 2, which have been already found, (and which are the ſame in both the foregoing calculations,) to wit, the four figures 1.259, for the baſis of the next operation, we ſhall obtain the value of the ſaid cube-root to a very great degree of exactneſs. This may be done in the manner following.

Let a be ſuppoſed $= 1.259$.

Then we ſhall have a^3 ($= \overline{1.259}|^3$) $=$ 1.995,616,979, and conſequently $2a^3$ ($= 2 \times$ 1.995,616,979) $=$ 3.991, 233,958, and $c + 2a^3$ ($= 2 +$ 3.991,233,958) $=$ 5.991,233,958, and $c - a^3$ ($=$ 2.000,000,000 — 1.995, 616,979) $=$ 0.004,383,021, and $\overline{c - a^3}| \times a$ ($=$ 0.004, 383,021 $\times$ 1.259) $=$ 0.005,518,223,439, and $\frac{\overline{c-a^3}| \times a}{c + 2a^3}$ ($= \frac{0.005,518,223,439}{5.991,233,958}$) $=$ 0.000,921,049,55, &c. Therefore $a + \frac{\overline{c-a^3}| \times a}{c + 2a^3}$ will be ($=$ 1.259 $+$ 0.000,921,049,55, &c,) $=$ 1.259,921,049,55, &c; which is exact in the firſt ten figures 1.259,921,049, the more accurate value of the cube-root of 2 being 1.259,921,049,89, which is greater than 1.259,921,049,55 by only the very ſmall quantity 0.000,000,000,34.

Alſo we ſhall have $4c$ ($= 4 \times 2$) $=$ 8.000,000,000, and $4c - a^3$ ($=$ 8.000,000,000 $-$ 1.995,616,979) $=$ 6.004,383,021, and $12a$ ($= 12 \times$ 1.259) $=$ 15.108, and $\frac{4c - a^3}{12a}$ ($= \frac{6.004,383,021}{15.108}$) $=$ 0.397,430,900,3, and $\sqrt{\frac{4c-a^3}{12a}}$ ($= \sqrt{}$0.397,430,700,3) $=$ 0.630,421,050,01. Therefore $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ will be ($= \frac{1.259}{2} +$ 0.630,421,050,01 $=$ 0.6295 $+$ 0.630,421,050,01) $=$ 1.259,921,050,01; which exceeds the more accurate value of the cube-root of 2, to wit,

wit, 1.259,921,049,89, by only the very ſmall quantity 0.000,000,000,12, which is ſtill leſs than the ſmall quantity 0.000,000,000,34.

Another Example of the Extraction of the Cube-root of a Number, by the ſame Method of Approximation.

Art. 10. Let it be propoſed to find, in inches and decimal parts of an inch, the ſide of a cube that is equal to the Engliſh meaſure called a gallon, which contains 231 cubick inches; or, in other words, let it be required to find the cube-root of the number 231.

Now, if we look along the column of cube numbers in the foregoing table, we ſhall find that 216 is the cube of 6, and that 343 is the cube of 7. Therefore we may conclude that the cube-root of the propoſed number 231 muſt be greater than 6, but leſs than 7. We will therefore take 6 for the value of a, or for the firſt near value of $\sqrt[3]{231}$, with which we are to begin our approximation.

Now, ſince c is, in this example, equal to 231, and a is $= 6$, we ſhall have a^3 ($= \overline{6}|^3$) $= 216$, and $2a^3 = 432$, and $c + 2a^3$ ($= 231 + 432$) $= 663$, and $c - a^3$ ($= 231 - 216$) $= 15$, and $\overline{c - a^3}| \times a$ ($= 15 \times 6$) $= 90$, and $\frac{\overline{c - a^3}| \times a}{c + 2a^3}$ ($= \frac{90}{663} = 0.13$. Therefore $a + \frac{\overline{c - a^3}| \times a}{c + 2a^3}$ will be ($= 6 + 0.13$) $= 6.13$; which is therefore the ſecond near value of the cube-root of 231.

Now let a be taken $= 6.13$, in order to obtain a third near value of the cube-root of 231.

Then we ſhall have a^3 ($= \overline{6.13}|^3$) $= 230.346,397$, and $2a^3$ ($= 2 \times 230.346,397$) $= 460.692,794$, and $c + 2a^3$

 ($=$

$(= 231 + 460.692,794) = 691.692,794$, and $c - a^3$ $(= 231.000,000 - 230.346,397) = 0.653,603$, and $\overline{c - a^3} \times a$ $(= 0.653,603 \times 6.13) = 4.006,586,39$, and $\frac{\overline{c - a^3} \times a}{c + 2a^3}$ $(= \frac{4.006,586,39}{691.692,794}) = 0.005,792,436$. Therefore $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be $(= 6.13 + 0.005,792,436) = 6.135,792,436$; which is therefore the third near value of the cube-root of 231, or of the length, in inches and decimal parts of an inch, of the side of a cube that contains an English gallon. Q. E. I.

This number 6.135,792,436, is exact in the first nine figures, 6.135,792,43, the more accurate value of the cube-root of 231 being, according to Dr. Halley, (from whose tract upon this subject in the Philosophical Transactions this example is taken,) 6.135,792,439,661,958, &c. Therefore the number of figures obtained exactly in this instance by means of the expression $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ is just triple of the number of figures that are contained in a, or 6.13, agreeably to what is observed above in art. 7.

Art. 11. The other expression of the third near value of the cube-root of 231, to wit, the expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, may be computed as follows.

Since c is $= 231$, and a is $= 6.13$, we shall have $4c$ $(= 4 \times 231) = 924$, and $\frac{a}{2}$ $(= \frac{6.13}{2}) = 3.065$, and $12a$ $(= 12 \times 6.13) = 73.56$, and a^3 $(= \overline{6.13}|^3) = 230.346,397$, and $4c - a^3$ $(= 924.000,000 - 230.346,397) = 693.653,603$, and $\frac{4c - a^3}{12a}$ $(= \frac{693.653,603}{73.56}) = 9.429,766,218,053,289,8$, and $\sqrt{\frac{4c - a^3}{12a}}$ $(= \sqrt{9.429,766,218,053,289,8}) = 3.070,792,441$. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be $(= 3.065$

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+ 3.070,792,441) = 6.135,792,441; or the third near value of the cube-root of 231, obtained by means of the expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, will be 6.135,792,441.

Q. E. I.

Art. 12. This number 6.135,792,441, obtained by means of the expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, exceeds the more accurate value of the cube-root of 231, to wit, 6.135,792,439, &c, by only the ſmall fraction 0.000,000,002; which is ſomewhat leſs than the ſmall fraction 0.000,000,003, by which the former number 6.135,792,436, that was obtained by means of the expreſſion $a + \frac{\overline{c-a^3}| \times a}{c+2a^3}$, falls ſhort of the ſaid more accurate value. But both theſe differences enter in the ſame place of decimal fractions, to wit, the ninth place, and therefore the ſmall difference of exactneſs in theſe two expreſſions $a + \frac{\overline{c-a^3}| \times a}{c+2a^3}$, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ is not worth attending to. But Dr. Halley gives the preference to the latter expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ on another account, to wit, becauſe he thinks the extraction of the ſquare-root of the fraction $\frac{4c-a^3}{12a}$ a leſs laborious operation than the diviſion of $\overline{c-a^3}| \times a$ by the great diviſor $c+2a^3$. His words are as follows. "And this Formula [the irra-" tional formula $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, though he uſes a ſome-" what different notation,] is deſervedly preferable to the " rational [or $a + \frac{\overline{c-a^3}| \times a}{c+2a^3}$,] upon the account of the " great diviſor, which is not to be managed without a " great deal of labour; whereas the extraction of the " ſquare-root proceeds much more eaſily, as manifold ex-" perience has taught me."

Thefe two examples are taken from a very ingenious and ufeful tract on this fubject, intitled, *A new, exact, and eafy Method of finding the Roots of Equations Generally, and that without any previous Reduction*; written by the celebrated Dr. Edmund Halley, and publifhed firft in the Philofophical Tranfactions for the month of May 1694, Number 210, and afterwards in the year 1708, in the fecond volume of the Collection of Mathematical and Philofophical Tracts, intitled *Mifcellanea Curiofa*, in three volumes octavo. See the 2d volume of the faid *Mifcellanea Curiofa*, pages 70, 71, 72, 73, 74, and 75.

A Third Example of the Extraction of the Cube-root of a Number, by the fame Method of Approximation.

Art. 13. This example fhall be that which is given by Mr. Raphfon in his *Analyfis Æquationum Univerfalis*, Problem 2d. It is to find the cube-root of the number 37,945.

Now, if we look along the column of cube-numbers in the foregoing table, we fhall find that 35,937 is the cube of 33, and that 39,304 is the cube of 34. Therefore, fince the propofed number 37,945 is greater than 35,937, but lefs than 39,304, it follows that the cube-root of 37,945 will be greater than 33, but lefs than 34, and confequently that the two firft figures of it will be 33.

Here then we have $c = 37{,}945$, and $a = 33$. Therefore a^3 will be $= 35{,}937$, and $2a^3$ will be $(= 2 \times 35{,}937) = 71{,}874$, and $c + 2a^3$ will be $(= 37{,}945 + 71{,}874) = 109{,}819$, and $c - a^3$ will be $(= 37{,}945 - 35{,}937) = 2008$, and $\overline{c - a^3} \times a$ will be $(= 2008 \times 33) = 66{,}264$, and confequently $\frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be $(= \frac{66{,}264}{109{,}819}) = 0.6033$.

Therefore

Therefore $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, or the ſecond near value of the cube-root of the propoſed number 37,945, will be (= 33 + 0.6033, or) 33.6033; of which number the five firſt figures 33.603 are exact, the more accurate value of the ſaid cube-root being 33.603,526,179,43, &c.

Now let us ſuppoſe a to be = 33.6033, or the ſecond near value of the cube-root of 37,945 that has been already found; and let us, in order to obtain a third near value of it, repeat the foregoing proceſs.

Then we ſhall have a^3 = 37,944.233,801,747,937, and $2a^3$ (= 2 × 37,944.233,801,747,937,) = 75,888.467,603,495,874, and $c + 2a^3$ (= 37,945.000,000,000,000 + 75,888.467,603,495,874) = 113,833.467,603,495,874, and $c - a^3$ (= 37,945.000,000,000,000 − 37,944.233,801,747,937) = 0.766,198,252,063, and $\overline{c - a^3} \times a$ (= 0.766,198,252,063 × 33.6033) = 25.746,789,723,548,607,9, and conſequently $\frac{\overline{c - a^3} \times a}{c + 2a^3}$ $\left(= \frac{25.746,789,723,548,607,9}{113,833.467,603,495,874}\right)$ = 0.000,226,179,437,95. Therefore $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be (= 33.6033 + 0.000,226,179,437,95,) = 33.603,526,179,437,95; that is, the third near value of the cube-root of c, or 37,945, that is obtained by means of the rational expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, will be 33.603,526,179,437,95; which I believe to be exact in the firſt fifteen figures 33.603,526,179,437,9, if not in the laſt, or ſixteenth, figure 5 likewiſe.

Art. 14. The other expreſſion of the third near value of the cube-root of 37,945, to wit, the irrational expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, may be computed as follows.

Since c is = 37,945, and a is = 33.6033, we ſhall have $\frac{a}{2}$ $\left(= \frac{33.6033}{2}\right)$ = 16.801,65, and $12a$ (= 12 × 33.6033) =

= 403.2396, and $4c$ (= 4 × 37,945) = 151,780, and a^3 (= $\overline{33.6033}|^3$) = 37,944.233,801,747,937, and $4c - a^3$ (= 151,780.000,000,000,000 — 37,944.233,801,747,937) = 113,835.766,198,252,063, and $\frac{4c - a^3}{12a}$ (= $\frac{113,835.766,198,252,063}{403.2396}$) = 282.303,043,149,165,069, and $\sqrt{\frac{4c - a^3}{12a}}$ (= $\sqrt{}$ 282.303,043,149,165,069) = 16.801, 876,179,437,96. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be (= 16.801,65 + 16.801,876,179,437,96) = 33.603,526,179, 437,96; that is, the third near value of the cube-root of c, or 37,945, that is obtained by means of the irrational expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, will be 33.603,526,179,437,96.

Art. 15. This number 33.603,526,179,437,96 muſt (if there has been no miſtake made in the calculation,) be ſomewhat greater than the true value of the cube-root of c, or 37,945; and the former number 33.603,526,179,437,95, obtained by means of the rational expreſſion $a + \frac{\overline{c - a^3}| \times a}{c + 2a^3}$, muſt be ſomething leſs than the ſaid true value. Theſe two numbers differ only by an unit in the laſt, or ſixteenth, figure. And hence it follows that the firſt ſixteen figures of the ſaid true value muſt be the ſame with thoſe of the leſſer of thoſe two numbers, or 33.603,526,179,437,95. Mr. Raphſon, however, computes it to be 33.603,526,179, 438,08. But I ſuſpect that his three laſt figures are not exact. But, whether they are exact or not, we may, at leaſt, conclude that, ſo far as theſe different calculations agree with each other, they muſt be exact, and conſequently that the firſt thirteen figures of the cube-root of the number 37,945 are 33.603,526,179,43.

A fourth

A Fourth Example of the Extraction of the Cube-root of a Number, by means of the foregoing Method of Approximation.

Art. 16. This example ſhall be one that is given by Monſieur de Lagney himſelf, in his book intitled, *Nouveaux Elements d'Arithmétique et d'Algébre*, which was publiſhed at Paris in Duodecimo, in the year 1697. It is to find the cube-root of the number 696,536,483,318,640,035,073, 641,037, which conſiſts of twenty ſeven figures, and may be expreſſed in the words following, to wit, 696 quadrillions, or fourth powers of a million, 536,483 trillions, or third powers of a million, 318,640 billions, or ſecond powers of a million, 035,073 millions, and 641,037 units.

This number is ſo great that it will be convenient to divide it into theſe two parts, 696,536,483,000,000,000,000,000,000 and 318,640,035,073,641,037, and to begin by ſeeking the cube-root of the firſt part, 696,536,483,000,000,000,000, 000,000.

Now this number 696,536,483,000,000,000,000,000,000 is = 696,536,483 × 1,000,000,000,000,000,000, or 696,536,483 × the cube of 1,000,000. Therefore its cube-root will be equal to 1,000,000 × the cube root of 696,536,483. Therefore, if we can find the cube-root of 696,536,483, we need only multiply it by 1,000,000, in order to obtain the cube root of 696,536,483,000,000,000, 000,000,000. We will therefore endeavour to find the cube-root of 696,536,483.

Now this number 696,536,483 is greater than 696,536,000, or than 696,536 × 1000, or than 696,536 × the cube of 10. Therefore the cube-root of 696,536,483 will be greater than the product of the multiplication of the cube root of 696,536 into 10. Therefore, if we can find the cube-root of 696,536, we need only multiply it by 10, in order to obtain the cube-root of 696,536,000, which will be ſomething

 leſs

lefs than the cube-root of 696,536,483, and may ferve as a bafis from which to begin our approximation to the true value of the faid cube-root. We will therefore now endeavour to find, to a fmall degree of exactnefs, the cube-root of 696,536.

Art. 17. Now, if we look into the foregoing table of the cubes of the natural numbers 1, 2, 3, 4, 5, 6, 7, &c, as far as 100, we fhall find that 681,472 is the cube of 88, and that 704,969 is the cube of 89. It follows therefore that the cube root of 696,536, (which is greater than 681,472, but lefs than 704,969,) muft be greater than 88, but lefs than 89, and confequently that the two firft figures of it muft be 88. Therefore the cube-root of 696,536,000 muft be greater than 88 × 10, or 880, but lefs than 89 × 10, or 890, and confequently the two firft figures of it will be 88. Therefore 880, being lefs than the cube-root of the number 696,536,000, will be lefs alfo than the cube-root of the number 696,536,483, which is greater than 696,536,000; but it will approach fufficiently near to it to enable us to begin a further approximation to it by means of the foregoing expreffions of Monfieur de Lagney.

Art. 18. Let us therefore fuppofe a to be $= 880$; and, in order to find a fecond near value of the cube-root of c, or 696,536,483, let us compute the expreffion $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$.

Here then we fhall have a^3 $(= \overline{880}|^3) = 681,472,000$, and $2a^3 = 1,362,944,000$, and $c + 2a^3$ $(= 696,536,483 + 1,362,944,000) = 2059,480,483$, and $c - a^3$ $(= 696,536,483 - 681,472,000) = 15,064,483$, and $\overline{c-a^3} \times a$ $(= 15,064,483 \times 880) = 13,256,745,040$, and confequently $\frac{\overline{c-a^3}\times a}{c+2a^3}$ $(= \frac{13,256,745,040}{2059,480,483}) = 6.436,9$. Therefore $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$ will be $(= 880 + 6.436,9) = 886.4369$; and confequently 886.4369 will be a fecond near value of the cube-root of the number c, or 696,536,483.

Art. 19.

Art. 19. Therefore (by what is ſhewn in art. 16,) 886.4369 × 1000,000, or 886,436,900 will be nearly equal to, but ſomewhat leſs than, the cube-root of the number 696,536,483,000,000,000,000,000,000, and, *à fortiori*, will be leſs than the cube-root of the propoſed number 696,536,483,318,640,035,073,641,037. And, as 886,436,900 is not much leſs than 886,437,000, it ſeems probable that 886,437,000 will likewiſe be ſomewhat leſs than the cube-root of the ſaid number 696,536,483,318,640,035,073,641,037. And ſo upon trial we ſhall find it to be. For the cube of 886,437,000 is 696,535,206,998,055,453,000,000,000, which is leſs than the ſaid propoſed number.

Art. 20. Now let a be ſuppoſed to be = 886,437,000, and let us endeavour to find a nearer value of the cube-root of the propoſed number 696,536,483,318,640,035,073,641,037, by computing the expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$.

Then, ſince c is = 696,536,483,318,640,035,073,641,037, and a is = 886,437,000, and conſequently a^3 is = 696,535,206,998,055,453,000,000,000, we ſhall have $\frac{a}{2}$ ($= \frac{886,437,000}{2}$) = 443,218,500, and

$4c$ (= 4 × 696,536,483,318,640,035,073,641,037)
= 2,786,145,933,274,560,140,294,564,148

and $4c - a^3$ (= 2,786,145,933,274,560,140,294,564,148,
— 696,535,206,998,055,453,000,000,000)
= 2,089,610,726,276,504,687,294,564,148, and $12a$ (= 12 × 886,437,000) = 10,637,244,000, and $\frac{4c-a^3}{12a}$ ($= \frac{2,089,610,726,276,504,687,294,564,148}{10,637,244,000}$) = 196,442,868,686,332,983, and $\sqrt{\frac{4c-a^3}{12a}}$ (= $\sqrt{}$ 196,442,868,686,332,983) = = 443,218,759. Therefore $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ will be (=

443,218,500 + 443,218,759) = 886,437,259; and consequently the cube-root of the proposed long number 696,536,483,318,640,035,073,641,037, will be very nearly equal to 886,437,259. Q. E. I.

Monsieur de Lagney determines this cube-root to be only 886,437,166. But this is owing, as I apprehend, to his intirely neglecting the last eighteen figures, 318,640, 035,073,641,037, of the proposed number, and consequently giving us only the cube-root of the number 696,536,483,000,000,000,000,000,000, which is less than the proposed number: whereas in the last operations of the foregoing process we took notice of all the figures of the proposed number, when we found the value of $4c$, and extracted the square-root of the fraction $\frac{4c-a^3}{12a}$.

Mr. de Lagney adds, as a proof of the great usefulness of this method of extracting cube-roots, that the most skilful Arithmetician would not be able to find the cube root of this long number, 696,536483,318640,035073,641037, to the same degree of exactness, or to nine places of figures, by the common method of extracting the cube-root, in the space of a whole month. See Monsieur de Lagney's *Nouveaux Eléments d'Arithmétique et d'Algébre*, page 307.

A SCHOLIUM.

Art. 21. This very useful method of approximating to the cube-roots, and other roots, of numbers was first published by Mr. de Lagney, at Paris, in the *Journal des Sçavants* for the 14th of May 1691, and afterwards was published again at greater length, and with a demonstration, in a separate tract in quarto, in the month of May of the following year 1692. But Mr. Joseph Raphson had published his *Analysis Æquationum Universalis*, (which contains

tains a general method of finding the roots of all ſorts of equations by approximation,) in the year 1690: and his method of approximation is not very different from this of Mr. de Lagney; and the ground, or principle, of it is exactly the ſame. So that, if Mr. de Lagney had ſeen Mr. Raphſon's *Analyſis Æquationum* before he had diſcovered his own method of approximation, it would have been eaſy for him to have deduced his own method from Mr. Raphſon's; and in that caſe it would have been candid in him to acknowledge that he had ſeen Mr. Raphſon's book, and had been led by it to the diſcovery of his own method. This, however, he has not done; at leaſt, not in his *Nouveaux Eléments d'Arithmétique et d'Algébre*, which is the only book of his that I have ſeen. And therefore I ſuppoſe he invented his method of approximation by his own efforts, and without having met with Mr. Raphſon's book.

Art. 22. It may further be obſerved, that Mr. Raphſon himſelf was not the firſt inventor of the method of reſolving equations by approximations of the kind he has made uſe of, that is, by approximations performed by transforming the original equation into another equation that involves in it the powers of the unknown difference between the firſt near value (already obtained, by conjecture or otherwiſe,) of the root of the firſt equation and its true value, and by reſolving the ſaid transformed equation in the manner of a ſimple equation, or by dropping all the terms that involve in them any higher powers of the ſaid unknown difference, or root of the ſecond equation, than its ſimple power. But this excellent method of diſcovering the roots of high equations had been found out by the great Sir Iſaac Newton more than twenty-four years before the publication of Mr. Raphſon's *Analyſis Æquationum Univerſalis*, to wit, in the year 1666, when he wrote his learned little tract, intitled *De Analyſi per æquationes numero terminorum infinitas*, which is printed in the *Commercium Epiſtolicum* of Mr John Collins and other Mathematicians, pages 67, 68, 69, &c. - - 93, of the 2d edition. This tract was firſt printed in the year 1712, in the firſt edition of the ſaid *Commercium Epiſtolicum.*

licum. But it had been ſhewn in manuſcript to Dr. Iſaac Barrow, and by him ſent to the ſaid Mr. John Collins, with Mr. Newton's leave, in the month of July 1669, and ſhewn to the Lord Viſcount Brouncker, (a great Mathematician of that time,) and, probably, to many other learned Mathematicians that were Fellows of the Royal Society, to which Mr. Collins was then a Secretary. And afterwards a part of it, containing a ſhort ſpecimen of Mr. Newton's method of reſolving equations by approximation, was publiſhed by Dr. Wallis in the 94th chapter of his Algebra in the year 1685, which was five years before the publication of Mr. Raphſon's *Analyſis Æquationum Univerſalis.* Whether this ſpecimen ſuggeſted to Mr. Raphſon the diſcovery of his method of approximating to the roots of equations, (which differs but little from that of Sir Iſaac Newton,) it is difficult to determine. He has not mentioned Newton's method in his treatiſe on this ſubject, though he was a great admirer of his genius, and ever ready to commend him: and therefore I am inclined to think that the above-mentioned ſpecimen of Newton's method of approximation was not the circumſtance that led him to the diſcovery of his own. But, whether it was or not, it is certain that the honour of priority with reſpect to this very uſeful invention is due to Sir Iſaac Newton.

Of the Ground, or Principle, of the Inveſtigation of the foregoing Expreſſions, invented by Monſieur de Lagney, *for Approximating to the Value of the Cube-root of a given Number.*

Art. 23. The inveſtigation of all the foregoing expreſſions invented by Monſieur de Lagney for approximating to the cube-root of a given number, when a firſt near value of the ſaid cube-root that is exact to one, or two, or more, places

places of decimal figures, is already known, is not difficult. It refults from the contemplation of the compound quantities that are equal to the cubes of a binomial quantity, (fuch as $a + b$,) and a refidual quantity, (fuch as $a - b$,) according as a, or the firft near value of $\sqrt[3]{c}$ which is already known, is lefs, or greater, than $\sqrt[3]{c}$; and therefore it ought properly to be divided into two parts, the one relating to the cafe in which a, or the firft near value of the cube-root of the given number c that is already known, is lefs than the cube-root of c, and the other relating to the cafe in which a, or the faid firft near value of $\sqrt[3]{c}$, is greater than the faid cube-root. The firft of thefe inveftigations, (by which we fhall alfo obtain Mr. Raphfon's approximation to the value of the faid cube-root, in the fame cafe, or when a is lefs than $\sqrt[3]{c}$,) is as follows.

An Inveftigation of the two Expreffions, $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, *and* $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, *given by Monfieur de* Lagney, *for a Second near Value of the Cube-root of a given Number* c, *when* a, *or a Firft near Value of it that is already known, is lefs than its true Value.*

Art. 24. Let z be put for the unknown difference by which a, or the firft near value of the cube root of the given number c, falls fhort of its true value; fo that $a + z$ fhall be $= \sqrt[3]{c}$.

Then will $\overline{a + z}|^3$ be $= c$. But $\overline{a + z}|^3$ is $= a^3 + 3aaz + 3azz + z^3$. Therefore $a^3 + 3aaz + 3azz + z^3$ will alfo be $= c$. This cubick equation is the foundation both of of Mr. Raphſon's and of Monſieur de Lagney's methods of approximating further to the true value of $\sqrt[3]{c}$.

Art. 25. Mr. Raphſon's approximation is obtained as follows. Since z is leſs, and uſually much leſs, than a, to wit, about a 10th or a 100th part of it, or, perhaps, ſtill leſs,) it follows that both $3azz$ and z^3 will be leſs, and uſually much leſs, than $3aaz$, and conſequently that $a^3 + 3aaz$ will be nearly equal to $a^3 + 3aaz + 3azz + z^3$, and therefore to c. Let them therefore be ſuppoſed to be equal to c. Then, ſince $a^3 + 3aaz$ are $= c$, we ſhall have $3aaz = c - a^3$, and conſequently $z = \frac{c - a^3}{3aa}$, which fraction conſiſts intirely of known quantities. Therefore $a + z$ will be $= a + \frac{c - a^3}{3aa}$, and $a + \frac{c - a^3}{3aa}$ will be a ſecond near value of $\sqrt[3]{c}$, or the cube-root of the given number c. Q. E. I.

This is Mr. Raphſon's approximation to the cube-root of c, when a is leſs than the ſaid cube-root; and it is the ſimpleſt and eaſieſt approximation that can well be imagined, and approaches very conſiderably beyond a to the true value of $\sqrt[3]{c}$. For it uſually gives us twice as many figures exact as we had before in a, or the firſt near value of $\sqrt[3]{c}$. And it is leſs operoſe, or difficult to compute, than Mr. de Lagney's firſt, or rational, expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$; becauſe it is eaſier to divide $c - a^3$ by $3aa$, or three times the ſquare of a, than, firſt, to multiply $c - a^3$ by a, and then to divide the product by $c + 2a^3$, which is a longer number than $3aa$. And for theſe reaſons Mr. Raphſon, in the Appendix to the ſecond edition of his *Analyſis Æquationum Univerſalis*, (which was publiſhed in the year 1697, ſeveral years after the publication of Mr. de Lagney's method of approximation,) declares that he continued to prefer his own approximation to thoſe of Mr. de Lagney, notwithſtanding their greater exactneſs.

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We will now proceed to investigate Mr. de Lagney's first, or rational, expression above-mentioned, in obtaining which Mr. Raphson's approximation is made use of as a necessary step.

Art. 26. Mr. de Lagney's first, or rational, expression, $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, is obtained by preserving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 + 3aaz + 3azz + z^3 = c$, or by supposing $a^3 + 3aaz + 3azz$ to be equal to c, and resolving the quadratick equation $a^3 + 3aaz + 3azz = c$, resulting from that supposition, in an imperfect, or inaccurate, manner, by proceeding as follows.

Since $a^3 + 3aaz + 3azz$ is $= c$, we shall have $3aaz + 3azz = c - a^3$, and (dividing both sides by $3aa + 3az$) $z = \frac{c - a^3}{3aa + 3az}$. Let us now substitute, instead of z, in the denominator of this fraction $\frac{c - a^3}{3aa + 3az}$, the near value of z already obtained by the resolution of the simple equation $a^3 + 3aaz = c$, to wit, the fraction $\frac{c - a^3}{3aa}$; and we shall thereby obtain the equation $z = \frac{c - a^3}{3aa + 3a \times \frac{c - a^3}{3aa}}$; which last quantity is equal to $\frac{c - a^3}{3aa + \frac{c - a^3}{a}}$ ($= \frac{c - a^3}{\frac{3a^3 + c - a^3}{a}} = \frac{c - a^3}{\frac{c + 2a^3}{a}} = \overline{c - a^3} \times \frac{a}{c + 2a^3}$) $= \frac{\overline{c - a^3} \times a}{c + 2a^3}$. Therefore z will be $= \frac{\overline{c - a^3} \times a}{c + 2a^3}$, and consequently $a + z$ will be $= a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$. Therefore the true value of $a + z$, or the cube-root of the given number c, will be nearly $= a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$.   Q. E. I.

 Art. 27.

Art. 27. This expreſſion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ of the ſecond near value of $\sqrt[3]{c}$, will always be leſs than its true value; as may be demonſtrated in the manner following.

Since $c - a^3$ is $= 3aaz + 3azz + z^3$, and $3aaz + 3azz + z^3$ is $= 3aaz + azz + 2azz + z^3$, it follows that $c - a^3$ will be $= 3aaz + azz + 2azz + z^3$, and conſequently will be greater than $3aaz + azz$. Therefore $\frac{c - a^3}{a}$ will be greater than $\frac{3aaz + azz}{a}$, or than $3az + zz$. But $\frac{c - a^3}{a}$ is $= 3a \times \frac{c - a^3}{3aa}$. Therefore $3a \times \frac{c - a^3}{3aa}$ will be greater than $3az + zz$; and conſequently (adding $3aa$ to both ſides,) $3aa + 3a \times \frac{c - a^3}{3aa}$ will be greater than $3aa + 3az + zz$. Therefore $\frac{c - a^3}{3aa + 3a \times \frac{c - a^3}{3aa}}$ will be leſs than $\frac{c - a^3}{3aa + 3az + zz}$. But $\frac{c - a^3}{3aa + 3a \times \frac{c - a^3}{3aa}}$ is $= \frac{\overline{c - a^3} \times a}{c + 2a^3}$, and $\frac{c - a^3}{3aa + 3az + zz}$ is $=$ the true value of z. Therefore $\frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be leſs than the true value of z. Therefore $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ will be leſs than the true value of $a + z$, or than $\sqrt[3]{c}$.

Q. E. D.

This expreſſion, $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, gives uſually three times as many figures of the value of $\sqrt[3]{c}$ exact as were given exactly by a, or the firſt near value of the ſaid cube-root. But in ſome caſes the figures which it gives exactly are only three times as many wanting one, and in ſome unfavourable caſes only three times as many wanting two, as were exact

exact in a; as we have seen in some of the foregoing examples.

Art. 28. Mr. de Lagney's second, or irrational, expression, $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, for the second near value of the cube-root of c, when a is less than the said cube-root, is obtained by preserving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 + 3aaz + 3azz + z^3 = c$, and resolving the equation thence resulting, to wit, the quadratick equation $a^3 + 3aaz + 3azz = c$, in an accurate manner. This may be done as follows.

If we suppose $a^3 + 3aaz + 3azz$ to be equal to c, we shall have $3aaz + 3azz = c - a^3$, and (dividing all the terms by $3a$) $az + zz = \frac{c - a^3}{3a}$. Therefore (adding $\frac{aa}{4}$ to both sides,) we shall have $\frac{aa}{4} + az + zz = \frac{c - a^3}{3a} + \frac{aa}{4}$ $(= \frac{4c - 4a^3}{12a} + \frac{3a^3}{12a} = \frac{4c - 4a^3 + 3a^3}{12a}) = \frac{4c - a^3}{12a}$, and (extracting the square-roots of both sides,) $\frac{a}{2} + z = \sqrt{\frac{4c - a^3}{12a}}$. Therefore z will be $= \sqrt{\frac{4c - a^3}{12a}} - \frac{a}{2}$, and $a + z$ will be $(= a + \sqrt{\frac{4c - a^3}{12a}} - \frac{a}{2}) = \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Therefore $\sqrt[3]{c}$ will be nearly equal to the same quantity $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Q. E. I.

Art. 29. This expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ must always be greater than the true value of $\sqrt[3]{c}$, or than the true value of $a + z$ in the original cubick equation $a^3 + 3aaz + 3azz + z^3 = c$.

For it is derived from a supposition that $a^3 + 3aaz +$

$3azz$ is equal to c, or to $a^3 + 3aaz + 3azz + z^3$, or is greater than it really is: from which it will neceffarily follow that the value of z deduced from that fuppofition muft be greater than its true value, and confequently that the value of $a + z$ deduced from that fuppofition, that is, the expreffion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be greater than the true value of $a + z$, or than $\sqrt[3]{c}$. Q. E. D.

Art. 30. The irrational expreffion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will approach a little nearer than the rational expreffion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ to the true value of $\sqrt[3]{c}$, becaufe it is obtained by refolving the quadratick equation $a^3 + 3aaz + 3azz = c$ accurately, whereas the rational expreffion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$ is obtained by refolving the fame quadratick equation inaccurately, by fubftituting $\frac{c - a^3}{3aa}$ inftead of z in the quantity $3az$ in the denominator of the fraction $\frac{c - a^3}{3aa + 3az}$ in art. 26. But the difference of the two expreffions in point of exactnefs is not confiderable; and the principal reafon for preferring the irrational expreffion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ to the rational expreffion $a + \frac{\overline{c - a^3} \times a}{c + 2a^3}$, is, that there is much lefs labour in extracting the fquare-root of the fraction $\frac{4c - a^3}{12a}$, than in dividing the numerator $\overline{c - a^3} \times a$ by the denominator $c + 2a^3$, when that denominator is a very long number.

An Investigation of the Two Expressions, $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, *given by Monsieur de* Lagney *for a Second near Value of the Cube-root of a given Number* c, *when* a, *or a First near Value of it that is already known, is greater than its true Value.*

Art. 31. Let z be put for the unknown difference by which a, or the first near value of the cube-root of the given number c, exceeds its true value; so that $a - z$ shall be $= \sqrt[3]{c}$.

Then will $\overline{a - z}|^3$ be $= c$. But $\overline{a - z}|^3$ is $= a^3 - 3aaz + 3azz - z^3$. Therefore $a^3 - 3aaz + 3azz - z^3$ will also be $= c$. This cubick equation is the foundation both of Mr. Raphson's and of Monsieur de Lagney's methods of approximating further to the true value of $\sqrt[3]{c}$.

Art. 32. Mr. Raphson's approximation is obtained as follows.

Since z is less, and usually much less, than a, (to wit, about a 10th, or a 100th, part of it, or, perhaps, still less,) it follows that both $3azz$ and z^3 will be less, and usually much less, than $3aaz$, and consequently that $a^3 - 3aaz$ will be nearly equal to $a^3 - 3aaz + 3azz - z^3$, and therefore to c. Let them therefore be supposed to be equal to c. Then, since $a^3 - 3aaz$ are $= c$, we shall have $a^3 = c + 3aaz$, and $3aaz = a^3 - c$, and consequently $z = \frac{a^3 - c}{3aa}$. Therefore $a - z$ will be $= a - \frac{a^3 - c}{3aa}$, and consequently the expression $a - \frac{a^3 - c}{3aa}$ will be a second near value of $\sqrt[3]{c}$, or the cube-root of the given number c. Q. E. I.

This

This is Mr. Raphſon's approximation to the cube-root of c, when a is greater than the ſaid cube-root; and it is the ſimpleſt and eaſieſt approximation that can well be imagined, and approaches much nearer than a to the true value of the ſaid cube-root. For it uſually gives us the value of the ſaid cube-root exact to twice as many decimal places of figures as were exact in a, or the firſt near value of the ſaid cube-root. And it is leſs operoſe, or difficult to compute, than Mr. de Lagney's firſt, or rational, expreſſion, $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$; becauſe it is eaſier to divide $c - a^3$ by $3aa$, or three times the ſquare of a, than, firſt, to multiply $a^3 - c$ by a, and then to divide the product by $c + 2a^3$, which is a longer number than $3aa$. And for theſe reaſons Mr. Raphſon always preferred it to Mr. de Lagney's approximations, notwithſtanding their greater exactneſs.

We will now proceed to inveſtigate Mr. de Lagney's firſt expreſſion above-mentioned, to wit, the rational expreſſion $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, in obtaining which Mr. Raphſon's approximation is made uſe of as a neceſſary ſtep.

Art. 33. Mr. de Lagney's firſt, or rational, expreſſion, $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, is obtained by preſerving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$, or by ſuppoſing $a^3 - 3aaz + 3azz$ to be equal to c, and reſolving the quadratick equation $a^3 - 3aaz + 3azz = c$, reſulting from that ſuppoſition; in an imperfect, or inaccurate, manner, by proceeding as follows.

Since $a^3 - 3aaz + 3azz$ is ſuppoſed to be equal to c, we ſhall have $a^3 + 3azz = c + 3aaz$, and $a^3 = c + 3aaz - 3azz$, and $a^3 - c = 3aaz - 3azz$, or $3aaz - 3azz = a^3 - c$, and (dividing both ſides of the equation by $3aa - 3az$,) $z = \frac{a^3 - c}{3aa - 3az}$. Mr. de Lagney then ſubſti-

tutes,

tutes, instead of z, in the denominator of the fraction $\frac{a^3 - c}{3aa - 3az}$, the near value of z already obtained by the resolution of the simple equation $a^3 - 3aaz = c$, to wit, the fraction $\frac{a^3 - c}{3aa}$, and thereby obtains the equation $z = \frac{a^3 - c}{3aa - 3a \times \overline{\frac{a^3 - c}{3aa}}}$; which last quantity is equal to $\frac{a^3 - c}{3aa - \overline{\frac{a^3 - c}{a}}}$ $\left(= \frac{a^3 - c}{\frac{3a^3 - \overline{a^3 - c}}{a}} = \frac{a^3 - c}{\frac{3a^3 - a^3 + c}{a}} = \frac{a^3 - c}{\frac{2a^3 + c}{a}} = \overline{a^3 - c} \times \frac{a}{c + 2a^3}\right) = \frac{\overline{a^3 - c} \times a}{c + 2a^3}$. Therefore z will be $= \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, and consequently $a - z$ will be $= a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$. Therefore the true value of $a - z$, or of the cube-root of the given number c, will be nearly $= a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$.

Q. E. I.

Art. 34. This expression, $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, of the second near value of $\sqrt[3]{c}$, will always be greater than its true value; as may be demonstrated in the manner following.

It has been shewn above in art. 31, that $a^3 - 3aaz + 3azz - z^3$ is $= c$. Therefore $a^3 + 3azz$ will be $= c + 3aaz + z^3$, and a^3 will be $= c + 3aaz - 3azz + z^3$, and $a^3 - c$ will be $= 3aaz - 3azz + z^3$. But $3aaz - 3azz + z^3$ is $= 3aaz - azz - 2azz + z^3$. Therefore $a^3 - c$ will be $= 3aaz - azz - 2azz + z^3$. But, because z is less than a, z^3 will be less than azz, and, *à fortiori*, less than $2azz$. Therefore $3aaz - azz - 2azz + z^3$ will be less than $3aaz - azz$; and consequently $a^3 - c$ (which is $= 3aaz - azz - 2azz + z^3$,) will be less than $3aaz - azz$. Therefore $\frac{a^3 - c}{a}$ will be less than $\frac{3aaz - azz}{a}$, or than $3az - zz$.

zz. But $\frac{a^3 - c}{a}$ is $= 3a \times \sqrt{\frac{a^3 - c}{3aa}}$. Therefore $3a \times \sqrt{\frac{a^3 - c}{3aa}}$ will be less than $3az - zz$. Therefore, if both these quantities be subtracted from $3aa$, the remainder $3aa - 3a \times \sqrt{\frac{a^3 - c}{3aa}}$ will be greater than the remainder $3aa - \overline{3az - zz}$, or than $3aa - 3az + zz$. Therefore $\frac{a^3 - c}{3aa - 3a \times \sqrt{\frac{a^3 - c}{3aa}}}$ will be less than $\frac{a^3 - c}{3aa - 3az + zz}$. But $\frac{a^3 - c}{3aa - 3a \times \sqrt{\frac{a^3 - c}{3aa}}}$ is $=$ $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$. Therefore $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$ will be less than $\frac{a^3 - c}{3aa - 3az + zz}$. But $\frac{a^3 - c}{3aa - 3az + zz}$ is equal to the value of z in the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$, or to its true value. Therefore $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$ will be less than the true value of z. And consequently $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ will be greater than the true value of $a - z$, or than the cube-root of c. Q. E. D.

Art. 35. The other, or irrational, expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, given by Mr. de Lagney for the second near value of the cube-root of c, when a is greater than the said cube-root, is obtained by preserving the term $3azz$, as well as the term $3aaz$, of the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$, obtained in art. 31, or by supposing the trinomial quantity $a^3 - 3aaz + 3azz$ to be equal to c, and resolving the quadratick equation $a^3 - 3aaz + 3azz = c$ in an accurate manner. This may be done as follows.

If we suppose $a^3 - 3aaz + 3azz$ to be $= c$, we shall have $a^3 + 3azz = c + 3aaz$, and $a^3 = c + 3aaz - 3azz$, and $a^3 - c = 3aaz - 3azz$, or $3aaz - 3azz = a^3 - c$. Therefore

Therefore (dividing both ſides of the equation by $3a$,) we ſhall have $az - zz = \frac{a^3 - c}{3a}$.

The compound quantity $az - zz$, which forms the left-hand ſide of this equation, is $= \overline{a - z} \times z$, and conſequently (by Euclid's Elements, Book 2d, Prop. 5,) muſt be leſs than the ſquare of $\frac{a}{2}$, or than $\frac{aa}{4}$. And conſequently the other ſide of the equation, or the quantity $\frac{a^3 - c}{3a}$, will alſo be leſs than $\frac{aa}{4}$. They may therefore both be ſubtracted from $\frac{aa}{4}$. Let them be ſo ſubtracted. And we ſhall then have $\frac{aa}{4} - \overline{az - zz} = \frac{aa}{4} - \frac{a^3 - c}{3a}$, or $\frac{aa}{4} - az + zz = \frac{aa}{4} - \frac{a^3 - c}{3a}$ $(= \frac{3a^3}{12a} - \frac{4a^3 - 4c}{12a} = \frac{3a^3 - \overline{4a^3 - 4c}}{12a} = \frac{3a^3 - 4a^3 + 4c}{12a}) = \frac{4c - a^3}{12a}$. Therefore the ſquare-root of the trinomial quantity $\frac{aa}{4} - az + zz$, will be equal to the ſquare-root of the fraction $\frac{4c - a^3}{12a}$. Now, if z could be of two different values, the one greater than $\frac{a}{2}$, and the other leſs than $\frac{a}{2}$, the trinomial quantity $\frac{aa}{4} - az + zz$ might have two ſquare-roots, to wit, $\frac{a}{2} - z$, and $z - \frac{a}{2}$. But, as z in the preſent problem is ſuppoſed to be much leſs than $\frac{a}{2}$, the latter of theſe ſquare-roots, to wit, $z - \frac{a}{2}$, cannot exiſt, and the other ſquare-root, $\frac{a}{2} - z$, will be the only one conſiſtent with the conditions of the Problem. Therefore we ſhall

 have

have $\frac{a}{2} - z = \sqrt{\frac{4c - a^3}{12a}}$, and consequently (adding z to both sides,) $\frac{a}{2} = \sqrt{\frac{4c - a^3}{12a}} + z$, and (subtracting $\sqrt{\frac{4c-a^3}{12a}}$ from both sides,) $z = \frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$. Therefore $a - z$ will be $= a - \overline{\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}}$ $(= a - \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}})$ $= \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$. Therefore the second near value of $\sqrt[3]{c}$ will be $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$. Q. E. I.

Art. 36. This expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will always be less than the true value of the cube-root of the given number c.

For, if we suppose z to increase continually from o till it becomes equal to $\frac{a}{2}$, the compound quantity $az - zz$, or $\overline{a - z} \times z$, will increase continually at the same time till it becomes equal to $\overline{\frac{a}{2}}^2$, or $\frac{aa}{4}$; and consequently the compound quantity $3a \times \overline{az - zz}$, or the compound quantity $3aaz - 3azz$, will increase continually till it becomes equal to $3a \times \frac{aa}{4}$, or $\frac{3a^3}{4}$. Therefore, when the compound quantity $3aaz - 3azz$ is equal to $a^3 - c$, the quantity z will be greater than when the compound quantity $3aaz - 3azz$ is equal to $a^3 - c - z^3$, which is less than $a^3 - c$; that is, the value of z in the quadratick equation $3aaz - 3azz = a^3 - c$ will be greater than the value of z in the cubick equation $3aaz - 3azz = a^3 - c - z^3$, or in the cubick equation $3aaz - 3azz + z^3 = a^3 - c$, or in the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$. But the value of z in the quadratick equation $3aaz - 3azz = a^3 -$

$a^3 - c$ is $\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$; and the value of z in the cubick equation $a^3 - 3aaz + 3azz - z^3 = c$ is the true value of z, or of the exceſs of a above $\sqrt[3]{c}$. Therefore the expreſſion $\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}$ will be greater than the true value of z. Therefore $a - \overline{\frac{a}{2} - \sqrt{\frac{4c - a^3}{12a}}}$, or $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be leſs than the true value of $a - z$, or than the cube-root of the given number c. Q. E. D.

Art. 37. The irrational *Formula* $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will approach a little nearer than the rational *Formula* $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ to the true value of the cube-root of c; becauſe the irrational *Formula* is derived from the quadratick equation $a^3 - 3aaz + 3azz = c$ by an accurate reſolution of it, and the rational *Formula* is derived from the ſame equation by an inaccurate reſolution of it. But the difference of exactneſs between theſe two expreſſions is not great, and either of them will uſually give us three times as many decimal figures of $\sqrt[3]{c}$ exact as were exact in a, or the firſt near value of it. But, when the given number c itſelf conſiſts of nine, or ten, or more figures, and alſo when a conſiſts of three, or four, figures, and conſequently a^3 conſiſts of nine, or ten, or more, figures, the irrational expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, will be found to be much eaſier to compute than the rational expreſſion $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, on account of the labour of dividing $\overline{a^3 - c} \times a$ by the long number $c + 2a^3$; and therefore Dr. Halley thinks it ought to be preferred to the other.

Art. 38. I have now given very full inveſtigations of the four expreſſions invented by Mr. de Lagney for the ſecond near value of the cube-root of a given number c, of which a firſt near value, denoted by the letter a, is already known, to wit, the four expreſſions $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{c+2a^3}}$, and $a - \frac{\overline{a^3-c}\times a}{c+2a^3}$, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$; of which the two firſt relate to the caſe in which a, or the firſt near value of the ſaid cube-root, is leſs than its true value; and the two laſt relate to the caſe in which a, or the firſt near value of the ſaid cube-root, is greater than its true value. And I have given demonſtrations of what is aſſerted concerning theſe four expreſſions in art. 5 and 6, to wit, that the firſt of them, or $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$, is always leſs than the true value of $\sqrt[3]{c}$, and that the ſecond of them, or $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, is always greater than the ſaid true value, and that the third expreſſion $a - \frac{\overline{a^3-c}\times a}{c+2a^3}$ is always greater than the ſaid true value, and that the fourth expreſſion $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ is always leſs than the ſaid true value. And the two firſt of theſe aſſertions have been confirmed upon trial in the examples given in art. 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20, of the extraction of the cube-roots of the numbers 2, 231, 37,945, and the long number 696,536,483,318,640,035,073,641,037; the ſaid cube-roots having been extracted by means of the two firſt expreſſions $a + \frac{\overline{c-a^3}\times a}{c+2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, which relate to the caſe in which a, or the firſt near value of $\sqrt[3]{c}$, is leſs than its true value. I will now therefore give an example, or two, of the extraction of the cube-roots of numbers by means of the two latter expreſſions

$a -$

$a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, which relate to the case in which a, or the first near value of $\sqrt[3]{c}$, is greater than its true value.

An Example of the Extraction of the Cube-root of a given Number, by means of Mr. de Lagney's *Third and Fourth Expressions* $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ *and* $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, *in which* a, *or the First near Value of* $\sqrt[3]{c}$, *is supposed to be greater than its true Value.*

Art. 39. Let it be required to find the cube-root of 2, which was extracted above in art. 8 and 9; and let a, or the first near value of the said cube-root, be 1.26, which is somewhat greater than its true value, which is 1.259,921, 049, &c.

Here then we have $c = 2$, and $a = 1.26$. Therefore a^3 will be $= 2.000,376$, and $2a^3$ will be $= 4.000,752$, and $c + 2a^3$ will be $(= 2 + 4.000,752) = 6.000,752$, and $a^3 - c$ will be $(= 2.000,376 - 2) = 0.000,376$, and $\overline{a^3 - c} \times a$ will be $(= 0.000,376 \times 1.26) = 0.000,473,76$, and consequently $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$ will be $(= \frac{0.000,473,76}{6.000,752}) = 0.000,078,950$. Therefore $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$, or the second near value of the cube-root of 2, will be $(= 1.260,000,000 - 0.000,078,950) = 1.259,921,050$; which is a little greater than its true value 1.259,921,049, &c, agreeably to what is asserted in art. 6, and demonstrated in art. 34.

And we shall have $4c$ $(= 4 \times 2) = 8.000,000$, and $\frac{a}{2}$ $(= \frac{1.26}{2}) = 0.63$, and $12a$ $(= 12 \times 1.26) = 15.12$, and

and $4c - a^3$ (= 8.000,000 — 2.000,376) = 5.999,624, and $\frac{4c-a^3}{12a}$ ($= \frac{5.999,624}{15.12}$) = 0.396,800,529,100,529,100,529, &c, and $\sqrt{\frac{4c-a^3}{12a}}$ (= $\sqrt{}$0.396,800,529,100,529,100,529, &c,) = 0.629,921,049,894,76, and $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$ (= 0.630,000,000,000,00 + 0.629,921,049,894,76) = 1.259, 921,049,894,76. Therefore the second near value of the cube-root of 2, obtained by the irrational expression $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$, will be 1.259,921,049,894,76; which is a little less than the true value of the said cube-root, agreeably to what is asserted in art. 6, and demonstrated in art. 36, the more accurate value of the said cube-root being 1.259,921, 049,894,873,164,76, &c.

Art. 40. The foregoing more accurate value of the cube-root of 2, to wit, 1.259,921,049,894,873,164,76, was obtained by taking 1.259,921,0 for a, or the first near value of the said cube-root, and computing the expression $\frac{a}{2} + \sqrt{\frac{4c-a^3}{12a}}$.

For, if a is = 1.259,921,0, or 1.259,921, we shall have $\frac{a}{2}$ = 0.629,960,5, and a^3 (= $\overline{1.259,921}|^3$) = 1.999,999, 762,390,486,961, and $4c$ (= 4 × 2) = 8.000,000,000, 000,000,000, and $4c - a^3$ = 8.000,000,000,000,000,000
— 1.999,999,762,390,486,961
= 6.000,000,237,609,513,039,
and $12a$ (= 12 × 1.259,921) = 15.119,052, and consequently $\frac{4c-a^3}{12a}$ ($= \frac{6.000,000,237,609,513,039}{15.119,052}$) = 0.396,850,294,423,850,982,125,069,746,436, and $\sqrt{\frac{4c-a^3}{12a}}$ (= $\sqrt{}$ 0.396850,294423,850982,125069,746436) = 0.629,960,549,894,873,164,76, &c. Therefore $\frac{a}{2}$ + $\sqrt{}$

$\sqrt{\frac{4c - a^3}{12a}}$ will be ($=$ 0.629,960,5 + 0.629,960,549,894, 873,164,76, &c,) $=$ 1.259,921,049,894,873,164,76, &c, which therefore will be a very near value of the cube-root of 2.

All the twenty-one figures of this number 1.259,921, 049,894,873,164,76, may be depended upon as exact, if no mistake has been made in computing the value of the expression $\sqrt{\frac{4c - a^3}{12a}}$; because *a*, or the first value of the cube-root of 2, to wit, 1.259,921,0, consists of eight figures which are all exact, and the number of figures that are exact in $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ is always triple, or triple wanting one or two figures, of the number of figures that are exact in *a*; as was observed in art. 7.

Art. 41. And in like manner in the second example, relating to the extraction of the cube-root of 231, if we take *a* equal to 6.14 (which is somewhat greater than the true value of the said cube-root,) instead of taking it equal to 6.13, (which is somewhat less than the said true value,) and compute the two expressions $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, we shall find the former of these expressions to be somewhat greater, and the latter of them to be somewhat less, than the true value of the said cube-root, agreeably to what is asserted in art. 6. These computations may be performed as follows.

If *a* is supposed to be $=$ 6.14, we shall have $\frac{a}{2}$ ($= \frac{6.14}{2}$) $=$ 3.07, and 12*a* ($=$ 12 $\times$ 6.14) $=$ 73.68, and a^3 ($= \overline{6.14}|^3$) $=$ 231.475,544, and $a^3 - c$ ($=$ 231.475,544 $-$ 231) $=$ 0.475,544, and $\overline{a^3 - c} \times a$ ($=$ 0.475,544 $\times$ 6.14) $=$ 2.919,840,16, and $2a^3$ ($=$ 2 $\times$ 231.475,544) $=$ 462.

462.951,088, and $c + 2a^3$ (= 231 + 462.951,088) = 693.951,088, and consequently $\frac{\overline{a^3 - c} \times a}{c + 2a^3}$ ($= \frac{2.919,840,16}{693.951,088}$) = 0.004,207,559. Therefore $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ will be (= 6.140,000,000 − 0.004,207,559) = 6.135,792,441; which is greater than the true value of the cube-root of 231, agreeably to what is asserted in art. 6, and demonstrated in art. 34, the said true value being only 6.135,792,439,661,958, &c. See above, art. 10, page 476.

And we shall have $4c$ (= 4 × 231) = 924, and $4c - a^3$ (= 924.000,000 − 231.475,544) = 692.524,456, and $\frac{4c - a^3}{12a}$ ($= \frac{692.524,456}{73.68}$) = 9.399,083,279,044,516,829, and $\sqrt{\frac{4c - a^3}{12a}}$ (= $\sqrt{9.399083,279044,516829}$) = 3.065,792,439,004. Therefore $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ will be (= 3.07 + 3.065,792,439,004,) = 6.135,792,439,004; which is less than the true value of the cube-root of 231, to wit, 6.135,792,439,661,958, &c, agreeably to what is asserted in art. 6, and demonstrated in art. 36.

Art. 42. These two examples are sufficient to illustrate and confirm what is asserted in art. 6, and demonstrated in art. 34 and 36, concerning the two expressions $a - \frac{\overline{a^3 - c} \times a}{c + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$; which are given by Monsieur de Lagney for a second near value of the cube-root of a given number c, when a, or the first near value of it, is greater than its true value. And with them I shall conclude the present tract.

End of the Tract on the Cubes of the Natural Numbers 1, 2, 3, 4, 5, 6, 7, *&c, and on Mr. de* Lagney's *Method of Extracting the Cube-roots of Numbers by Approximation.*

A GENE-

A

GENERAL METHOD

OF

EXTRACTING THE ROOTS OF NUMBERS

BY

APPROXIMATION;

INVENTED BY

MONSIEUR DE LAGNY,

A MEMBER OF THE ROYAL ACADEMY OF SCIENCES AT PARIS,

AND PUBLISHED IN THE YEAR 1697,

IN HIS

Nouveaux Eléments d'Arithmétique et d'Algêbre.

A GENERAL METHOD OF EXTRACTING THE ROOTS OF NUMBERS BY *APPROXIMATION.*

Article 1. IN the foregoing Tract I have inferted Monfieur de Lagny's Method of Extracting the Cube-roots of Numbers by Approximation, and have both given full inveftigations of it, and illuftrated it by feveral examples that clearly prove its great utility. But Mr. de Lagny did not confine this method to the extraction of the cube-roots of numbers, but extended it to the extraction of their fifth roots, and their feventh roots, and all higher roots of them whatfoever. This he did by purfuing the fame principle by which he had before been enabled to find his approximations to the cube-root of a given number, to wit, by confidering the conftitution of the compound quantity that is equal to any given power of a binomial quantity, (fuch as $a + b$,) or of a refidual quantity, (fuch as $a - b$,) and fubftituting the fum, or difference, between a, the firft near value of the root fought, (which is fuppofed to be already known,) and z, its unknown difference from the true value of the faid root fought, inftead of the faid true value itfelf in the original equation derived from the conditions of the Problem, and then refolving the new equation, refult-

 ing

ing from fuch fubftitution, as if it were a quadratick equation, or neglecting all the terms of it which involve any higher powers of its root, or the difference z, than the fquare. This method I fhall now endeavour to explain in the folutions of the two following Problems.

PROBLEM I.

Art. 2. Let N be any propofed number whatfoever, and m any propofed whole number whatfoever; and let a be a known number that is nearly equal to, but lefs than, the mth root of the given number N. It is required to find a fecond near value of the faid mth root of the given number N, that fhall approach much nearer to it than a, or the former near value of it that is already known.

SOLUTION.

Let z be put for the unknown difference between a, the firft near value of the mth root of the given number N, and the true value of the faid number. Then, fince a is fuppofed to be lefs than $\sqrt[m]{N}$, it follows that $a + z$ will be $= \sqrt[m]{N}$, and confequently that $\overline{a+z}|^m$ will be $=$ N.

But, by the binomial theorem in the firft and fimpleft cafe of it, to wit, the cafe of integral powers, $\overline{a+z}|^m$ will be $=$ the feries $a^m + m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z^2$

+

$+ m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3}z^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4}z^4 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times a^{m-5}z^5$ + &c, continued to $m + 1$ terms; or, if, for the ſake of brevity, we ſubſtitute the capital letters A, B, C, D, E, F, &c, inſtead of the ſeveral numeral co-efficients $1, m, m \times \frac{m-1}{2}, m \times \frac{m-1}{2} \times \frac{m-2}{3}, m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, and $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, &c, reſpectively, $\overline{a+z}|^m$ will be = the ſeries $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2 + Da^{m-3}z^3 + Ea^{m-4}z^4 + Fa^{m-5}z^5$ + &c, continued to $m + 1$ terms. Therefore the ſaid ſeries $Aa^m + Ba^{m-1}z + Ca^{m-2}z^2 + Da^{m-3}z^3 + Ea^{m-4}z^4 + Fa^{m-5}z^5$ + &c, continued to $m + 1$ terms, will be = N. This is the original equation, by the ſolution of which we are to find a near value of z, and conſequently a ſecond near value of $a + z$, or $\sqrt[m]{N}$.

Art. 3. Now, ſince z is leſs than a, and uſually much leſs, being about a 10th or 100th part of it, or ſome ſtill leſſer part of it, it is evident that all the terms in the aforeſaid ſeries that involve zz, and z^3, and z^4, and the following powers of z, will be leſs, and uſually much leſs, than the term $Ba^{m-1}z$, which involves only the ſimple power of z. And therefore, if all the ſaid terms of the ſeries be neglected or omitted, the two firſt terms alone, to wit, $Aa^m + Ba^{m-1}z$, will be nearly equal to the given number N; and conſequently (if we ſubtract Aa^m from both ſides of the equation,) we ſhall have $Ba^{m-1}z$ nearly = $N - Aa^m$, and (dividing both ſides by Ba^{m-1}) z nearly =

$= \frac{N - A a^m}{B a^{m-1}}$, or (becaufe A is $= 1$, and B is $= m$) z nearly $= \frac{N - a^m}{m a^{m-1}}$; which fraction may be derived from the known quantities N and a by the operations of Multiplication, Subtraction, and Divifion. This therefore is an approximation to the true value of z, and confequently $a + \frac{N - a^m}{m a^{m-1}}$ will be an approximation to the true value of $a + z$, or of $\sqrt[m]{N}$, or will be a fecond near value of it that will approach nearer to it than a, or the firft near value of it which was already known. But it will evidently be fomewhat greater than the true value of $\sqrt[m]{N}$; becaufe it arofe from a fuppofition that $A a^m + B a^{m-1} z$ were equal to the whole feries of which they are only the two firft terms, or that they were greater than they really were.

This quantity, $a + \frac{N - a^m}{m a^{m-1}}$, is the expreffion given by Mr. Raphfon for the fecond near value of the mth root of the given number N. And it is a very ufeful approximation: for it ufually gives us twice as many figures of the true value of $\sqrt[m]{N}$ exact as were exact in a, or the firft near value of the faid mth root. And it is evidently the moft fimple and eafy approximation to the value of the faid mth root that can well be imagined.

Art. 4. But Mr. de Lagny, being defirous of finding at once a ftill nearer value of the mth root of the number N, retains the third term $C a^{m-2} z^2$, as well as the two firft terms $A a^m$ and $B a^{m-1} z$, of the feries $A a^m + B a^{m-1} z + C a^{m-2} z^2 + D a^{m-3} z^3 + E a^{m-4} z^4 + F a^{m-5} z^5 +$ &c, (which is equal to N,) and thereby converts the original equation

equation $A a^m + B a^{m-1} z + C a^{m-2} z^2 + D a^{m-3} z^3 + E a^{m-4} z^4 + F a^{m-5} z^5 +$ &c, $= N$ into a quadratick equation, to wit, the equation $A a^m + B a^{m-1} z + C a^{m-2} z^2 = N$, inftead of converting it (as Mr. Raphfon does,) into the fimple equation $A a^m + B a^{m-1} z = N$. And this quadratick equation he refolves firft imperfectly, or inaccurately, by fubftituting in one of its terms, inftead of z, the inaccurate value of z already found by the refolution of the fimple equation $A a^m + B a^{m-1} z = N$, to wit, the fraction $\frac{N - a^m}{m a^{m-1}}$, (by which fubftitution the quadratick equation is reduced to a fimple equation,) and then refolving the faid fimple equation thereby obtained; which produces a fecond value of z that is nearer than the former value $\frac{N - a^m}{m a^{m-1}}$ to its true value. And this gives him a rational expreffion for the value of $a + z$, or the fecond near value of $\sqrt[m]{N}$. And then he refolves the fame quadratick equation, $A a^m + B a^{m-1} z + C a^{m-2} z^2$, accurately, by the common methods of refolving quadratick equations; which produces a furd, or irrational, expreffion for the value of z, and confequently another furd, or irrational, expreffion for the value of $a + z$, or for the fecond near value of $\sqrt[m]{N}$. Thefe refolutions of the faid quadratick equation $A a^m + B a^{m-1} z - C a^{m-2} z^2 = N$, may be performed in the following manner.

Art. 5. Since $A a^m + B a^{m-1} z + C a^{m-2}$ is $= N$, we fhall have $B a^{m-1} z + C a^{m-2} z^2 = N - A a^m$; that is, is, $z \times \mathrm{B}a^{m-1} + z \times \mathrm{C} \times a^{m-2}z$ will be $= \mathrm{N} - \mathrm{A}a^m$, or $z \times \overline{\mathrm{B}a^{m-1} + \mathrm{C} \times a^{m-2}z}$ will be $= \mathrm{N} - \mathrm{A}a^m$. Therefore (dividing both ſides of the equation by the compound quantity $\mathrm{B}a^{m-1} + \mathrm{C} \times a^{m-2}z$) we ſhall have $z = \frac{\mathrm{N} - \mathrm{A}a^m}{\mathrm{B}a^{m-1} + \mathrm{C}a^{m-2}z}$.

Now let $\frac{\mathrm{N} - \mathrm{A}a^m}{\mathrm{B}a^{m-1}}$, or $\frac{\mathrm{N} - a^m}{ma^{m-1}}$, (which has already been ſhewn to be nearly equal to z,) be ſubſtituted inſtead of z in the ſecond term $\mathrm{C}a^{m-2}z$ of the denominator of the fraction laſt obtained, to wit, the fraction $\frac{\mathrm{N} - \mathrm{A}a^m}{\mathrm{B}a^{m-1} + \mathrm{C}a^{m-2}z}$.

And we ſhall have $z = \frac{\mathrm{N} - \mathrm{A}a^m}{\mathrm{B}a^{m-1} + \mathrm{C}a^{m-2} \times \overline{\frac{\mathrm{N} - a^m}{ma^{m-1}}}}$, or (becauſe A is $= 1$, and B is $= m$,)

$$z = \frac{\mathrm{N} - a^m}{ma^{m-1} + \mathrm{C}a^{m-2} \times \overline{\frac{\mathrm{N} - a^m}{ma^{m-1}}}}; \text{ which is}$$

$$= \frac{\mathrm{N} - a^m}{\frac{m^2a^{2m-2} + \mathrm{C}a^{m-2} \times \overline{\mathrm{N} - a^m}}{ma^{m-1}}}$$

$$= \frac{\mathrm{N} - a^m}{\frac{m^2a^{2m-1} + \mathrm{CN}a^{m-2} - \mathrm{C}a^{2m-2}}{ma^{m-1}}} =$$

$$\overline{\mathrm{N} - a^m} \times \frac{ma^{m-1}}{m^2a^{2m-2} + \mathrm{CN}a^{m-2} - \mathrm{C}a^{2m-3}} =$$

$$= \frac{\overline{N - a^m} \times m a^{m-1}}{m^2 a^{2m-2} + C N a^{m-2} - C a^{2m-2}}$$

$= \frac{\overline{N - a^m} \times 2m a^{m-1}}{2m^2 a^{2m-2} + 2C N a^{m-2} - 2C a^{2m-2}} =$ (becaufe C is $= m \times \frac{m-1}{2}$, and confequently 2C is $= m \times \overline{m-1}$, which therefore may be fubftituted for it,)

$$\frac{\overline{N - a^m} \times 2m a^{m-1}}{2m^2 a^{2m-2} + m \times \overline{m-1} \times N a^{m-2} - m \times \overline{m-1} \times a^{2m-2}}$$

= (if we divide both the numerator and the denominator by m,) $\frac{\overline{N - a^m} \times 2a^{m-1}}{2m a^{2m-2} + \overline{m-1} \times N \times a^{m-2} - \overline{m-1} \times a^{2m-2}}$

$$= \frac{\overline{N - a^m} \times 2a^{m-1}}{\overline{2m - \overline{m-1}} \times a^{2m-2} + \overline{m-1} \times N \times a^{m-2}}$$

$$= \frac{\overline{N - a^m} \times 2a^{m-1}}{\overline{2m - m + 1} \times a^{2m-2} + \overline{m-1} \times N \times a^{m-2}}$$

$= \frac{\overline{N - a^m} \times 2a^{m-1}}{\overline{m+1} \times a^{2m-2} + \overline{m-1} \times N \times a^{m-2}} =$ (if we multiply both the numerator and denominator into a,)

$\frac{\overline{N - a^m} \times 2a^{m}}{\overline{m+1} \times a^{2m-1} + \overline{m-1} \times N \times a^{m-1}} =$ (by dividing both the numerator and the denominator of this fraction by a^{m-1},) $\frac{\overline{N - a^m} \times 2a}{\overline{m+1} \times a^{m} + \overline{m-1} \times N}$, or $\frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} + a^{m}}$.

Therefore z will be $= \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^{m}}$, and confequently $a + z$ will be $= a + \frac{2a \times \overline{N - a^m}}{\overline{m - 1} \times N + \overline{m + 1} \times a^m}$, or $a + \frac{2a \times \overline{N - a^m}}{\overline{m - 1} \times N + \overline{m + 1} \times a^m}$ will be a ſecond near value of $a + z$, or of the mth root of the propoſed number N.

Q. E. I.

Art, 6. The quadratick equation mentioned above in art. 4, to wit, the equation $A a^m + B a^{m-1} z + C a^{m-2} z^2 = N$, or (becauſe A is $= 1$, and B is $= m$, and C is $= m \times \frac{m-1}{2}$,) the equation $a^m + m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z^2 = N$, may be accurately reſolved in the manner following.

By doubling both ſides of this equation we ſhall have $2a^m + 2ma^{m-1} z + m \times \overline{m - 1} \times a^{m-2} zz = 2N$; and, by multiplying both ſides of this equation into aa, we ſhall have $2a^{m+2} + 2ma^{m+1} z + m \times \overline{m - 1} \times a^m \times zz = 2a^2 N$; and, by ſubtracting $2a^{m+2}$ from both ſides, we ſhall have $2ma^{m+1} z + m \times \overline{m - 1} \times a^m \times zz = 2aa N - 2a^{m+2}$; and, by dividing both ſides of this laſt equation by $m \times \overline{m - 1} \times a^m$, we ſhall have $\frac{2ma^{m+1} z}{m \times \overline{m-1} \times a^m} + zz = \frac{2aa N - 2a^{m+2}}{m \times \overline{m-1} \times a^m}$, or $\frac{2a^{m+1} z}{\overline{m-1} \times a^m} + zz = \frac{2aa N - 2a^{m+2}}{m \times \overline{m-1} \times a^m}$, or (becauſe $\frac{a^{m+1}}{a^m}$ is $= a$,) $\frac{2az}{m - 1} +$

$zz =$

$zz = \frac{2aaN - 2a^{m+2}}{m \times \overline{m-1} \times a^{m}}$, or $\frac{2a}{m-1} \times z + zz = \frac{2N - 2a^{m}}{m \times \overline{m-1} \times a^{m-2}}$;

or $\frac{2a}{m-1} \times z + zz = \frac{2 \times \overline{N - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}$.

Now let the square of $\frac{a}{m-1}$ (which is half the co-efficient of z in the term $\frac{2az}{m-1}$) be added to both sides of this equation. And we shall have $\overline{\frac{a}{m-1}}|^{2} + \frac{2a}{m-1} \times z + zz = \overline{\frac{a}{m-1}}|^{2} + \frac{2 \times \overline{N - a^{m}}}{m \times \overline{m-1} \times a^{m-2}} = \frac{aa}{\overline{m-1}|^{2}} + \frac{2 \times \overline{N - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}$; or, if, for the sake of brevity, we put P for the quantity $\frac{aa}{\overline{m-1}|^{2}} + \frac{2 \times \overline{N - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}$, we shall have $\overline{\frac{a}{m-1}}|^{2} + \frac{2a}{m-1} \times z + zz = P$. Therefore, (extracting the square-roots of both sides,) we shall have $\frac{a}{m-1} + z = \sqrt{P}$, and $z = \sqrt{P} - \frac{a}{m-1}$. Therefore $a + z$, or $\sqrt[m]{N}$, will be $= a + \sqrt{P} - \frac{a}{m-1}$, or $a + \sqrt{\frac{aa}{\overline{m-1}|^{2}} + \frac{2 \times \overline{N - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}} - \frac{a}{m-1}$. Q. E. I.

Art. 7. This irrational expression $a + \sqrt{\frac{aa}{\overline{m-1}|^{2}} + \frac{2 \times \overline{N - a^{m}}}{m \times \overline{m-1} \times a^{m-2}}} - \frac{a}{m-1}$, or $a + \sqrt{P} - \frac{a}{m-1}$, will approach somewhat nearer to the true value of $\sqrt[m]{N}$ than the former, or rational, expression,

$a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$; becaufe it proceeds from the accurate refolution of the quadratick equation $a^m + ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2}zz = N$, whereas the rational expreffion $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ was derived from an inaccurate refolution of the fame quadratick equation. But the difference of the two expreffions, in point of exactnefs, is not great; and either of them will ufually give us three times as many decimal figures of the true value of $\sqrt[m]{N}$ exact as were exact in a, or the preceeding near value of it.

Examples of the Extraction of the Roots of Numbers by Means of the foregoing Expreffions.

EXAMPLE I.

Art. 8. Let it be required to find the cube-root of the number 2, having 1.259 for a, or the firft near value of the faid cube-root.

Here N is = 2, and m is = 3, and confequently $m - 1$ is (= 3 − 1) = 2, and $m + 1$ is (= 3 + 1) = 4, and the expreffion $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ becomes $= a + \frac{2a \times \overline{N - a^3}}{2N + 4a^3}$, or $a + \frac{a \times \overline{N - a^3}}{N + 2a^3}$, or $a + \frac{\overline{N - a^3} \times a}{N + 2a^3}$, or, (if we fubftitute the fmall letter c inftead of the capital letter N,) N,) $a + \frac{\overline{c - a^3}| \times a}{c + 2a^3}$; which is the firſt, or rational, expreſſion, given in the preceeding tract for the cube-root of the number c. And as a is ſuppoſed to be $= 1.259$, this expreſſion $a + \frac{\overline{c - a^3}| \times a}{c + 2a^3}$ will be $= 1.259 + \frac{\overline{2 - \overline{1.259}|^3}| \times 1.259}{2 + 2 \times \overline{1.259}|^3}$ $= 1.259 + \frac{\overline{2 - 1.995,616,979}| \times 1.259}{2 + 2 \times 1.995,616,979} = 1.259 +$ $\frac{0.004,383,021 \times 1.259}{2 + 3.991,233,958} = 1.259 + \frac{0.005,518,223,439}{5.991,233,958} = 1.259$ $+ 0.000,921,049,55$, &c $= 1.259,921,049,55$, &c. Therefore $1.259,921,049,55$, &c, will be a near value of the cube-root of 2. Q. E. I.

See the preceeding Tract, page 474.

Art. 9. And if we compute this cube-root by means of the irrational expreſſion $a + \sqrt{P} - \frac{a}{m - 1}$, or $a - \frac{a}{m - 1} +$ $\sqrt{P}$, or $a - \frac{a}{m - 1} + \sqrt{\frac{aa}{\overline{m - 1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m - 1} \times a^{m-2}}}$, we ſhall find that this expreſſion will, upon making the proper ſubſtitutions, co-incide with the irrational expreſſion given for the cube-root of a given number c in the foregoing tract, to wit, the expreſſion $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, and conſequently give the ſame value of the cube root of 2, as was obtained in the foregoing tract by means of that expreſſion.

For, ſince m is $= 3$, we ſhall have $m - 1 = 2$, and $\overline{m - 1}|^2 = 4$, and $m \times \overline{m - 1}$ $(= 3 \times 2) = 6$, and $m - 2$ $(= 3 - 2) = 1$, and $a^{m-2} = a$, and conſequently $\frac{aa}{\overline{m - 1}|^2} = \frac{aa}{4}$, and $\frac{2 \times \overline{N - a^m}}{m \times \overline{m - 1} \times a^{m-2}} = \frac{2 \times \overline{N - a^3}}{6 \times a} = \frac{N - a^3}{3a}$,

and

and P, or $\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}$ ($= \frac{aa}{4} + \frac{N - a^3}{3a} = \frac{3a^3}{12a} + \frac{4N - 4a^3}{12a} = \frac{3a^3 + 4N - 4a^3}{12a}$) $= \frac{4N - a^3}{12a}$, or, (if we substitute c instead of N, in order to adopt the notation used in the foregoing tract,) $P = \frac{4c - a^3}{12a}$, and $a - \frac{a}{m-1} + \sqrt{P}$ ($= a - \frac{a}{2} + \sqrt{P} = \frac{a}{2} + \sqrt{P}$) $= \frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$; which is the irrational expression given in the foregoing tract for the second near value of the cube-root of the number c. So that the general, irrational, expression $a - \frac{a}{m-1} + \sqrt{P}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}}$, which is obtained by the general investigation of the mth root of the given number N in the present tract, agrees perfectly with the particular, irrational, expression $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$, which was obtained in the foregoing tract, by the particular investigation of the cube-root of the given number c.

And, since c is $= 2$, and a is $= 1.259$, we shall have a^3 ($= \overline{1.259}|^3$) $= 1.995,616,979$, and $4c - a^3$ ($= 8.000,000,000 - 1.995,616,979$) $= 6.004,383,021$, and $12a$ ($= 12 \times 1.259$) $= 15.108$, and $\frac{4c - a^3}{12a}$ ($= \frac{6.004,383,021}{15.108}$) $= 0.397,430,700,3$, and $\sqrt{\frac{4c - a^3}{12a}}$ ($= \sqrt{0.397430,7003}$) $= 0.630,421,050,01$, and $\frac{a}{2}$ ($= \frac{1.2590}{2}$) $= 0.6295$, and $\frac{a}{2} + \sqrt{\frac{4c - a^3}{12a}}$ ($= 0.6295 + 0.630,421,050,01$) $= 1.259,$

921,050,01. Therefore 1.259,921,050,01 will be a second near value of the cube-root of 2. Q. E. I.

The first eight figures 1.259,921,0, of this number 1.259,921,050,01, are exact, the more accurate value of the cube-root of 2 being 1.259,921,049,894,873,164,76, &c, of which I believe all the figures to be exact. See above, pages 502 and 503.

EXAMPLE II.

Art. 10. Let it be required to find the fifth root of the number 2, which the celebrated *Vieta*, or *Monsieur Viete*, has found to be 1.148,697.

Here m is $= 5$, and consequently $m - 1$ is $= 4$, and $m + 1$ is $= 6$, and the first general expression $a + \frac{2a \times \overline{N - a^m}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ will be $(= a + \frac{2a \times \overline{N - a^5}}{4N + 6a^5}) = a + \frac{a \times \overline{N - a^5}}{2N + 3a^5}$, or $a + \frac{\overline{N - a^5} \times a}{2N + 3a^5}$; and the other general expression $a - \frac{a}{m-1} + \sqrt{P}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}}$, will be $(= a - \frac{a}{4} + \sqrt{\frac{aa}{16} + \frac{2 \times \overline{N - a^5}}{5 \times 4 \times a^3}} = \frac{3a}{4} + \sqrt{\frac{aa}{16} + \frac{N - a^5}{10a^3}} = \frac{3a}{4} + \sqrt{\frac{10a^5}{160a^3} + \frac{16N - 16a^5}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{10a^5 + 16N - 16a^5}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{16N - 6a^5}{160a^3}}) = \frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$. We must therefore compute one of the two expressions $a + \frac{a \times \overline{N - a^5}}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, in order to obtain a

second

ſecond near value of the fifth root of N, or 2, when we ſhall have firſt found a, or a firſt near value of the ſaid fifth root that is leſs than its true value, to a ſmall degree of exactneſs.

Art. 11. The value of a, or the firſt approximation to the fifth root of 2, may be found in the following manner.

The fifth root of 2 is the ſecond of ſix quantities in continued geometrical proportion, of which 1 is the firſt, or leaſt, and 2 is the ſixth, or greateſt. Now the exceſs of the ſecond of theſe proportionals above the firſt, is neceſſarily leſs than the fifth part of the exceſs of the greateſt above the leaſt, becauſe the exceſſes increaſe in the ſame proportion as the terms themſelves. The exceſs of the greateſt of theſe ſix terms, to wit, 2, above the leaſt of them, or 1, is 1; of which the fifth part is $\frac{1}{5}$, or $\frac{2}{10}$, or 0.2. Therefore the exceſs of the ſecond term above the firſt will be leſs than $\frac{2}{10}$, or 0.2; and conſequently the ſecond term itſelf will be leſs than $1 + \frac{2}{10}$, or than 1.2. Let us therefore ſuppoſe this ſecond term, or the fifth root of 2, which we are ſeeking, to be equal to 1.1; and let us raiſe this number to the fifth power, in order to try how nearly it approaches to the truth.

Now, if we raiſe 1.1 to the fifth power, or multiply it four times into itſelf, we ſhall find that $\overline{1.1}|^5$ is $=$ 1.61051; which is conſiderably leſs than 2. Therefore 1.1 muſt be conſiderably leſs than the true value of the 5th root of 2. But we have ſeen that the ſaid true value muſt be leſs than 1.2. Let us therefore ſuppoſe it to be equally diſtant from 1.1 and 1.2, or to be $=$ 1.15, and try whether this will not be pretty near the truth.

Now $\overline{1.15}|^5$ is $=$ 2.011,357,187,5; which is a little bigger than 2. Therefore 1.15 muſt be ſomething greater than the 5th root of 2. But the difference can be but ſmall.

We

We will therefore, in the next place, ſuppoſe the ſaid 5th root to be = 1.14, and raiſe this number to the fifth power, in order to diſcover whether the ſaid fifth power will be greater, or leſs, than 2, and conſequently whether 1.14 will be greater, or leſs, than the fifth root of 2.

Now, if we multiply 1.14 four times ſucceſſively into itſelf, we ſhall find that $\overline{1.14}|^5$ is = 1.925,414,582,4; which is ſomewhat leſs than 2. Therefore the 5th root of 2 will be greater than 1.14, but leſs than 1.15; and therefore 1.14 will be a very convenient firſt near value of the 5th root of 2, and will be very fit to be made the baſis of a further approximation to the true value of the ſaid 5th root, by ſubſtituting it inſtead of *a* in either of the two expreſſions $a + \frac{\overline{N - a^5}| \times a}{2N + 3a^5}$ and $\frac{3a}{4} + \sqrt{\left(\frac{8N - 3a^5}{80a^3}\right)}$, which have been derived from the foregoing Problem.

Art. 12. Now, if we ſuppoſe *a* to be = 1.14, we ſhall have aa = 1.2996, and a^3 = 1.481,544, and a^5 = 1.925,414,582,4, and conſequently N — a^5 (= 2.000,000,000,0 — 1.925,414,582,4) = 0.074,585,417,6, and $\overline{N - a^5}| \times a$ (= 0.074,585,417,6 × 1.14) = 0.085,027,376,064, and 2N (= 2 × 2) = 4, and $3a^5$ (= 3 × 1.925,414,582,4) = 5.776,243,747,2, and 2N + $3a^5$ (= 4 + 5.776,243,747,2) = 9.776,243,747,2, and $\frac{\overline{N - a^5}| \times a}{2N + 3a^5}$ $\left(= \frac{0.085,027,376,064}{9.776,243,747,2}\right)$ = 0.008,697,34. Therefore $a + \frac{\overline{N - a^5}| \times a}{2N + 3a^5}$ will be (= 1.14 + 0.008,697,34) = 1.148,697,34; and conſequently 1.148,697,34 will be very nearly equal to the 5th root of 2. Q. E. I.

This number 1.148,697,34 agrees with that found by *Vieta*, to wit, 1.148,697, in all its ſeven figures, but is carried to two more figures.

The other expreſſion $\frac{3a}{4} + \sqrt{\left(\frac{8N - 3a^5}{80a^3}\right)}$, may be computed as follows.

Since a is $= 1.14$, we ſhall have $3a$ $(= 3 \times 1.14) =$ 3.42, and $\frac{3a}{4}$ $(= \frac{3.42}{4}) = 0.855$, and $8N$ $(= 8 \times 2) =$ 16, and $3a^5$ $(= 3 \times 1.925,414,582,4) = 5.776,243,747,2$, and $8N - 3a^5$ $(= 16.000,000,000,0 - 5.776,243,747,2)$ $= 10.223,756,252,8$, and $80a^3$ $(= 80 \times 1.481,544) =$ $118.523,520$, and $\frac{8N - 3a^5}{80a^3}$ $(= \frac{10.223,756,252,8}{118.523,520}) = 0.086,$ $259,303,240,40$, and $\sqrt{\frac{8N - 3a^5}{80a^3}}$ $(= \sqrt{0.086259,303240,}$ $40) = 0.293,699,34$, and conſequently $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$ $(= 0.855 + 0.293,699,34) = 1.148,699,34$. Therefore the fifth root of 2 will be very nearly equal to 1.148,699,34.

Q. E. I.

EXAMPLE III.

Art. 13. Let it be required to find the 5th root of the number 307,68282,11067,15625.

In order to find a firſt near value of the 5th root of this long number, we may begin by comparing it with the fifth powers of the numbers 10, 100, and 1000, and the following powers of 10. Now the fifth power of 10 is 100,000; which is very much leſs than the propoſed number: and the fifth power of 100 is 10,000,000,000; which is alſo much leſs than the propoſed number: and the fifth power of 1000 is 1000,000,000,000,000; which is alſo leſs than the propoſed number: but the fifth power of 10,000 is 100,000,000,000,000,000,000; which is greater than the propoſed number. We may therefore conclude that 1000 muſt be leſs, and that 10,000 muſt be greater than the fifth root

root of the propoſed number. Therefore the fifth root of the ſaid number muſt be of an intermediate magnitude between 1000 and 10,000.

Further, the propoſed number 307,68282,11067,15625 is greater than 307,00000,00000,00000, or than $307 \times \overline{1000}|^5$. Therefore the fifth root of the ſaid number will be greater than the product of the multiplication of the fifth root of 307 by 1000. We will therefore inquire what is the fifth root of 307.

Now the fifth power of 2 is 32, and the fifth power of 3 is 243, and the fifth power of 4 is 1024. Therefore the fifth power of 4 is much greater than 307, and the fifth power of 3 is a little leſs than 307. We may therefore conclude that the fifth root of 307 will be much leſs than 4, and a little greater than 3; and conſequently the fifth root of the number 307,00000,00000,00000 will be much leſs than 4×1000, or 4000, and a little greater than 3×1000, or 3000. We may therefore reaſonably conjecture that the fifth root of the propoſed number 307,68282,11067,15625, will be nearly equal to 3100. And accordingly, if we raiſe this number 3100 to the fifth power, we ſhall find the ſaid power of it to be = 286,29151,00000,00000, which is pretty nearly equal to, but ſomewhat leſs than, the propoſed number 307,68282,11067,15625. Therefore 3100 will be a proper number to make the baſis of a further approximation to the true value of the fifth root of the ſaid propoſed number, by means of either of the two expreſſions above-mentioned, to wit, $a + \frac{\overline{N - a^5}| \times a}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$.

Art. 14. Here then we have N = 307,68282,11067,15625, and a = 3100, and conſequently a^5 = 286,29151,00000,00000, and $3a^5$ (= 3 × 286,29151,00000,00000) = 858,87453,00000,00000, and 2N (= 2 × 307,68282,11067,15625) = 615,36564,22134,31250, and $2N + 3a^5$

(= 615,36564,22134,31250 + 858,87453,00000,00000) = 1474,24017,22134,31250, and N − a^5 (= 307,68282, 11067,15625 — 286,29151,00000,00000) = 21,39131, 11067,15625, and $\overline{N - a^5} \times a$ (= 21,39131,11067,15625 × 3100) = 66313,06443,08184,37500, and $\frac{\overline{N - a^5} \times a}{2N + 3a^5}$ (= $\frac{66313,06443,08184,37500}{1474,24017,22134,31250}$) = 44.98. Therefore $a + \frac{\overline{N - a^5} \times a}{2N + 3a^5}$ will be (= 3100 + 44.98) = 3144.98, which will therefore be nearly equal to the fifth root of the propofed number 307,68282,11067,15625. Q. E. I.

The three firft figures, 314, of this number 3144.98, or, rather, the five firft figures of it, 3144.9, are exact, the error being only in the fixth figure 8, which ought to be a 9 inftead of an 8. For the exact root of the propofed number 307,68282,11067,15625 is 3144.999,999, *ad infinitum*, or the whole number 3145, as will appear by raifing the faid number 3145 to the fifth power.

Art. 15. The other, or irrational, expreffion for the fecond near value of the fifth root of this number 307,68282, 11067,15625, to wit, the expreffion $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, may be computed as follows.

Since a is = 3100, we fhall have $3a$ (= 3 × 3100) = 9300, and $\frac{3a}{4}$ (= $\frac{9300}{4}$) = 2325, and a^3 = 2,97910,00000, and a^5 = 286,29151,00000,00000, and $80a^3$ (= 80 × 2,97910,00000) = 238,32800,00000, and 8N (= 8 × 307,68282,11067,15625) = 2461,46256,88537,25000, and $3a^5$ = 858,87453,00000,00000, and 8N — $3a^5$ (= 2461, 46256,88537,25000 — 858,87453,00000,00000) = 1602, 58803,88537,25000, and $\frac{8N - 3a^5}{80a^3}$ (= $\frac{1602,58803,88537,25000}{238,32800,00000}$) = 672429.6091, and $\sqrt{\frac{8N - 3a^5}{803^3}}$ (= $\sqrt{672429.6091}$)

820.01,

820.01, and consequently $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$ (= 2325 + 820.01) = 3145.01. Therefore 3145.01 will be a second near value of the fifth root of the proposed number 307,68282,11067,15625. Q. E. I.

Art. 16. If we should chuse to find the fifth root of this number 307,68282,11067,15625, by means of Mr. Raphson's expression for its value, to wit, the expression $a + \frac{N - a^m}{a^{m-1}}$, or $a + \frac{N - a^5}{5a^4}$, (which certainly has the merit of being much simpler, and easier to be remembered, than either of Mr. de Lagny's expressions, and likewise much easier to compute,) the computation will be as follows.

Since a is = 3100, we shall have a^4 = 9235,21000,00000, and a^5 = 286,29151,00000,00000, and $5a^4$ = 46176,05000,00000, and $N - a^5$ (= 307,68282,11067,15625 — 286,29151,00000,00000) = 21,39131,11067,15625, and $\frac{N - a^5}{5a^4}$ $\left(= \frac{21,39131,11067,15625}{46176,05000,00000}\right)$ = 46. Therefore $a + \frac{N - a^5}{5a^4}$ will be (= 3100 + 46) = 3146; which therefore will be the second near value of the fifth root of the proposed number 307,68282,11067,15625, obtained by Mr. Raphson's approximation. Q. E. I.

A SCHOLIUM.

This last near value of the fifth root of the said proposed number, which has been obtained by Mr. Raphson's approximation, is greater than its true value, 3145, by only an unit, or the 3145th part of the said true value. So that this very simple method of approximating to the roots of numbers

bers may be juſtly conſidered as extremely uſeful as well as eaſy. And, if this proceſs were to be repeated, by taking 3146 for the value of a, and ſuppoſing $a - z$ to be equal to $\sqrt[5]{N}$, (a, or 3146, being ſomewhat greater than the true value of the fifth root which we are in ſearch of,) and by computing the expreſſion $a - \frac{N - a^5}{5a^4}$, that would reſult from that ſuppoſition, this ſecond proceſs would double the number of figures that are exact in a, or 3146, or give us about four times the number of figures that were exact in 3100, or the former value of a; which is more than is done by either of the two expreſſions, $a + \frac{\overline{N - a^5} \times a}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, given us by Mr. *de Lagny*: So that two ſteps of Mr. *Raphſon*'s method of approximation are more than equivalent to one ſtep of Mr. *de Lagny*'s method. It may therefore be doubted, whether Mr. *Raphſon*'s method is not, upon the whole, to be preferred to Mr. *de Lagny*'s, as Mr. *Raphſon* himſelf always thought it to be. For he tells us in the Appendix to the ſecond edition of his excellent Treatiſe, intitled, *Analyſis Æquationum Univerſalis*, (which ſecond edition was publiſhed in the year 1697, ſeven years after the firſt edition of it, and five or ſix years after the publication of *Monſieur de Lagny*'s method,) that he himſelf had had thoughts of reſolving the quadratick equation $A a^m + B a^{m-1} z + C a^{m-2} z^2 = N$, or $a^m + m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z^2 = N$, in the imperfect manner adopted by Mr. *Lagny*, in order to obtain his rational value of z, to wit, $\frac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, to wit, by ſubſtituting in the term $m \times \frac{m-1}{2} \times a^{m-2} z^2$ of the ſaid quadratick equation, inſtead of z, the value of z already obtained by the reſolution of the ſimple equation $a^m + m a^{m-1} z = N$, to wit, the fraction

fraction $\frac{N - a^m}{ma^{m-1}}$, (by which substitution the said quadratick equation would be converted into the following simple equation, $a^m + ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2}z \times \overline{\frac{N - a^m}{ma^{m-1}}} = N$,) and then resolving the simple equation thence resulting, to wit, the simple equation $a^m + ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2}z \times \frac{N - a^m}{ma^{m-1}} = N$, or $ma^{m-1}z + m \times \frac{m-1}{2} \times a^{m-2} \times \overline{\frac{N - a^m}{ma^{m-1}}} \times z = N - a^m$, or $z \times \overline{ma^{m-1} + m \times \frac{m-1}{2} \times a^{m-2} \times \overline{\frac{N - a^m}{ma^{m-1}}}} = N - a^m$, in the usual way, or by the single operation of Division, which would give us z (=

$$\frac{N - a^m}{ma^{m-1} + m \times \frac{m-1}{2} \times a^{m-2} \times \overline{\frac{N - a^m}{ma^{m-1}}}}$$

$$= \frac{N - a^m}{ma^{m-1} + \frac{m-1}{2} \times a^{m-2} \times \overline{\frac{N - a^m}{a^{m-1}}}}$$

$$= \frac{2 \times \overline{N - a^m}}{2ma^{m-1} + \overline{m-1}\, a^{m-2} \times \overline{\frac{N - a^m}{a^{m-1}}}}$$

$$= \frac{2 \times \overline{N - a^m}}{\frac{2ma^{2m-2} + \overline{m-1} \times a^{m-2} \times \overline{N - a^m}}{a^{m-1}}}$$

=

$$= \frac{2 \times \overline{N - a^m}}{\dfrac{2m a^{2m-2} + \overline{m-1} \times a^{m-2} \times N - m a^{2m-2} + a^{2m-2}}{a^{m-1}}}$$

$$= \frac{2 \times \overline{N - a^m}}{\dfrac{m a^{2m-2} + a^{2m-2} + \overline{m-1} \times a^{m-2} \times N}{a^{m-1}}}$$

$$= \frac{2a \times \overline{N - a^m}}{\dfrac{m a^{2m-1} + a^{2m-1} + \overline{m-1} \times a^{m-1} \times N}{a^{m-1}}}$$

$$= \frac{2a \times \overline{N - a^m}}{m a^m + a^m + \overline{m-1} \times N} = \frac{2a \times \overline{N - a^m}}{\overline{m+1} \times a^m + \overline{m-1} \times N}$$

$= \dfrac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, and consequently $a + z = a + \dfrac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$; which is Mr. *de Lagny*'s rational expression for the second near value of the *m*th root of N. Mr. *Raphson* (I say,) tells us that he himself had had thoughts of resolving the quadratick equation $a^m + m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z^2 = N$ in this manner, in order to obtain a value of z somewhat nearer to the truth than the fraction $\frac{N - a^m}{m a^{m-1}}$, which he had obtained by the resolution of the simple equation $a^m + m a^{m-1} z = N$; but that he did not think proper to adopt this method, because he thought his own method of approximation, (which he published in the first edition of his *Analysis Æquationum Universalis*, in the year 1690, and which is derived from the simple equation $a^m + m a^{m-1} z = N$,) the easiest and clearest, and, upon the whole, the best, and fittest for practice, that could be followed. His words, in his Appendix, relating

relating to this ſubject, are as follows. *An Dominus* de Lagny *Librum meum unquam viderat, néc-ne, prorsùs neſcio. Quibúſce modis non ſolùm ſua methodus, ſed et etiàm aliæ quam-plurimæ, eodem prorsùs proceſſu, et perpetuâ inde derivatâ graduum ſcalâ, inveniri poſſint, hujúſce Appendicis eſt oſtendere; ídque, quàm poſſimus, breviſſimè.*

Ipſe equidem de gradatìm inferendis (quas priùs rejeceram in Theoremate Vietæo,) *poteſtatibus olìm cogitavi: Sed tamen non proſecutus fui; utpote qui methodum meam, harum omnium fundamentalem, veluti facillimam ſemper exiſtimavi. Subſequenti proceſſu earum omnium inventionem indagare cuilibet liceat.* See Mr. Raphſon's *Analyſis Æquationum Univerſalis*, Edition 2d, 1697, page 49. And again, in page 55, he concludes his Appendix with theſe words: *Innumeras etiàm alias methodos et abbreviationes* (novarum *quidem* methodorum *nomine inſigniendas,) adinvenire liceat; quæ tamen omnia fundamentali huic ſuperiorum poteſtatum imprimis rejectionis methodo, poſteáque gradatìm retinendarum, innitantur. Noſtram tamen ſimpliciſſimam fore et facillimam, cuivis pateat.*

EXAMPLE IV.

Art. 17. Let it be required to find the ſeventh root of the number 34,487,717,467,307,513,182,492,153,794,673; which, Mr. *Bonnycaſtle*, of the Royal Military Academy at Woolwich, in his *Scholar's* Guide to Arithmetick, page 189, tells us, is 32017.

This number muſt, in the firſt place, be compared with the ſeventh powers of 10, 100, 1000, &c, to which it approaches neareſt, in order to know between which two of thoſe numbers 10, 100, 1000, &c, its ſeventh root will lie. Now the ſeventh power of 10 is 10,000,000, which is very much leſs than the ſaid propoſed number; and the ſeventh power of 100 is 100,000,000,000,000, which is likewiſe much leſs than the ſaid propoſed number; and the ſeventh power of 1000 is 1000,000,000,000,000,000,000, which is likewiſe leſs than the ſaid propoſed number; and the ſeventh power of 10,000 is 10,000,000,000,000,000,000,000,000,000, or 1 with twenty-eight cyphers annexed to it; which is likewiſe leſs than the ſaid propoſed number, which conſiſts of 32 figures. But the ſeventh power of 100,000 is 100,000, 000,000,000,000,000,000,000,000,000,000, or 1 with 35 cyphers annexed to it, and is therefore greater than the ſaid propoſed number, which conſiſts of only 32 figures. Therefore 10,000 will be leſs, and 100,000 will be greater, than the ſeventh root of the ſaid propoſed number.

Further, the propoſed number 34,487,717,467,307,513, 182,492,153,794,673, is greater than the number 34480, 000,000,000,000,000,000,000,000,000, or than 3448 × 10,000,000,000,000,000,000,000,000,000, or than 3448 × the ſeventh power of 10,000: but the difference between them is not great. Therefore the ſeventh root of the ſaid propoſed number will be greater than 10,000 times the ſeventh root of 3448: but the difference between them will not be great. And conſequently, if we can find the ſeventh root of the number 3448 exact to two, or three, places of figures, we need only multiply the ſaid ſeventh root by 10,000, in order to obtain the ſeventh root of 34480,000, 000,000,000,000,000,000,000,000, exact to two or three places of figures. And, when we have obtained the ſaid near value of the ſeventh root of the number 34480,000, 000,000,000,000,000,000,000,000, the ſaid near value will likewiſe be a near value of the ſeventh root of the propoſed number 34487,717,467,307,513,182,492,153,794,673, and will be leſs than the true value of the ſeventh root of the ſaid number, and therefore will ſerve as a convenient baſis of a further approximation to the true value of the ſeventh root of the ſaid propoſed number, by means of one of Monſieur *de Lagny*'s two expreſſions found above in the ſolution of the foregoing Problem. We muſt therefore now endeavour to find a near value of the ſeventh root of the number

number 3448. Now this may be done in the manner following.

Art. 18. The ſeventh power of the number 2 is 128, and the ſeventh power of 3 is 2187, and the ſeventh power of 4 is 16384. Therefore, ſince 3448 is greater than 2187, or the ſeventh power of 3, but is much leſs than 16384, or the ſeventh power of 4, it follows that the ſeventh root of 3448 muſt be greater than 3, but much leſs than 4. We may therefore reaſonably conjecture, that it will be nearly equal to $3\frac{1}{5}$, or 3.2. And accordingly, upon trial, we ſhall find it to be ſo. For, if we raiſe 3.2 to its ſeventh power, we ſhall find the ſaid power to be = 3435.973,836,8; which is leſs than 3448, but very nearly equal to it. Therefore 3.2 is a very near firſt value of the ſeventh root of the number 3448; and conſequently 3.2 × 10,000, or 32000, will be a very near firſt value of the number 34480,000, 000,000,000,000,000,000,000,000, and therefore will be alſo a pretty near firſt value of the propoſed number 34487,717,467,307,513,182,492,153,794,673. We will therefore ſuppoſe a to be = 32000, and proceed, upon that ſuppoſition, to compute the two expreſſions given in the ſolution of the foregoing Problem, for a ſecond value of the ſeventh root of the ſaid propoſed number that ſhall approach nearer than a, or 32000, to its true value. Theſe computations will be as follows.

Art. 19. The firſt, or rational, expreſſion, given in the ſolution of the foregoing Problem, for the ſecond near value of the mth root of any propoſed number N, is $a + \frac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$. Now, when m is = 7, this expreſſion will be $\left(= a + \frac{\overline{N - a^7} \times 2a}{\overline{7-1} \times N + \overline{7+1} \times a^7} = a + \frac{\overline{N - a^7} \times 2a}{6N + 8a^7}\right) = a + \frac{\overline{N - a^7} \times a}{3N + 4a^7}$.

Now, ſince N is, in this caſe, = 34,487,717,467,307,

 513,

513,182,492,153,794,673, and a is = 32000, we ſhall have a^7 (= $\overline{32000}|^7$) = 34,359,738,368,000,000,000,000,000,000,000,000, and N — a^7
(= 34,487,717,467,307,513,182,492,153,794,673
— 34,359,738,368,000,000,000,000,000,000,000)
= 127,979,099,307,513,182,492,153,794,673, and
$\overline{N - a^7}| \times a$ (= 127,979,099,307,513,182,492,153,794,673 × 32000) = 4,095,331,177,840,421,839,748,921,429,536,000, and 3N (= 3 × 34,487,717,467,307,513,182,492,153,794,673) = 103,463,152,401,922,539,547,476,461,384,019, and $4a^7$
(= 4 × 34,359,738,368,000,000,000,000,000,000,000)
= 137,438,945,472,000,000,000,000,000,000,000, and $3N + 4a^7$
(= 103,463,152,401,922,539,547,476,461,384,019
+ 137,438,945,472,000,000,000,000,000,000,000)
= 240,902,097,873,922,539,547,476,461,384,019, and
$\frac{\overline{N - a^7}| \times a}{3N + 4a^7}$ (= $\frac{4,095,331,177,840,421,839,748,921,429,436,000}{240,902,097,873,922,539,547,476,461,384,019}$ =, nearly, $\frac{4,095,331}{240,902}$) = 16.99998. Therefore $a + \frac{\overline{N - a^7}| \times a}{3N + 4a^7}$ will be (= 32000 + 16.9998) = 32016.9998, and conſequently the ſecond near value of the ſeventh root of the propoſed number 34,487,717,467,307,513,182,492,153,794,673 will be 32016.9998. Q. E. I.

This number is true in all the figures but the laſt, which ought to be a 9 inſtead of an 8, the true value of the ſeventh root of the ſaid propoſed number being 32016.999,999,999,999, &c, *ad infinitum*, or the whole number 32017.

Art. 20. The ſecond, or irrational, expreſſion, given in the ſolution of the foregoing Problem, for the ſecond near value of the mth root of any propoſed number N, is a —

$$\frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}}$$; which, when m is

= 7,

$= 7$, (as is the caſe in the preſent example,) is $\left(= a - \frac{a}{7-1} + \sqrt{\frac{aa}{\overline{7-1}|^2} + \frac{2 \times \overline{N - a^7}}{7 \times \overline{7-1} \times a^5}} = a - \frac{a}{6} + \sqrt{\frac{aa}{6|^2} + \frac{2 \times \overline{N-a^7}}{7 \times 6 \times a^5}} = \frac{5a}{6} + \sqrt{\frac{aa}{36} + \frac{N - a^7}{7 \times 3 \times a^5}} = \frac{5a}{6} + \sqrt{\frac{aa}{36} + \frac{N-a^7}{21a^5}} = \frac{5a}{6} + \sqrt{\frac{21a^7}{21 \times 36a^5} + \frac{36N - 36a^7}{21 \times 36a^5}} = \frac{5a}{6} + \sqrt{\frac{7a^7}{7 \times 36a^5} + \frac{12N - 12a^7}{7 \times 36a^5}} = \frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{7 \times 36a^5}}\right) = \frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{252a^5}}$. This expreſſion may be computed as follows.

Since N is = 34,487,717,467,307,513,182,492,153,794, 673, and a is = 32000, we ſhall have $5a$ (= 5 × 32000) = 160,000, and $\frac{5a}{6}$ $\left(= \frac{160,000}{6}\right)$ = 26,666.666,666, &c, and a^5 $(= \overline{32000}|^5)$ = 33,554,432,000,000,000,000,000, and $252a^5$ (= 252 × 33,554,432,000,000,000,000,000) = 8,455,716,864,000,000,000,000,000, and a^7 $(= \overline{32,000}|^7)$ = 34,359,738,368,000,000,000,000,000,000,000, and $5a^7$ (= 5 × 34,359,738,368,000,000,000,000,000,000,000) = 171,798,691,840,000,000,000,000,000,000,000, and 12N (= 12 × 34,487,717,467,307,513,182,492,153,794,673) = 413,852,609,607,690,158,189,905,845,536,076, and $12N - 5a^7$ (= 413,852,609,607,690,158,189,905,845,536, 076 − 171,798,691,840,000,000,000,000,000,000,000) = 242,053,917,767,690,158,189,905,845,536,076, and $\frac{12N - 5a^7}{252a^5}$ $\left(= \frac{242,053,917,767,690,158,189,905,845,536,076}{8,455,716,864,000,000,000,000,000}\right)$ = 28,626,067.033799, and $\sqrt{\frac{12N - 5a^7}{252a^5}}$ (= √ 28,626,067. 033799) = 5350.333,35. Therefore $\frac{5a}{6} + \sqrt{\frac{12N - 5a^7}{252a^5}}$ will be (= 26,666.666,66, &c, + 5350.333,35) = 32017. 000,01 ; and conſequently 32017.000,01 will be the ſecond near

near value of the seventh root of the proposed number 34,487,717,467,307,513,182,492,153,794,673. Q. E. I.

This number 32017.000,01 is exact in the first nine figures, 32017.0000, and errs only in the 10th figure, 1, which ought to be a cypher, 0, instead of a 1, because the true value of this seventh root is 32017.000,000,000, &c, *ad infinitum*, or the whole number 32017.

Art. 21. If we seek the value of this seventh root by Mr. Raphson's expression $a + \frac{N - a^m}{m a^{m-1}}$, or $a + \frac{N - a^7}{7a^6}$, the computation will be as follows.

Since a is = 32,000, we shall have a^6 (= $\overline{32,000}^6$) = 1,073,741,824,000,000,000,000,000,000, and $7a^6$ (= 7 × 1,073,741,824,000,000,000,000,000,000) = 7,516,192,768,000,000,000,000,000,000, and a^7 (= $\overline{32,000}^7$) = 34,359,738,368,000,000,000,000,000,000,000, and $N - a^7$ (= 34,487,717,467,307,513,182,492,153,794,673 − 34,359,738,368,000,000,000,000,000,000,000) = 127,979,099,307,513,182,492,153,794,673, and $\frac{N - a^7}{7a^6}$ $\left(= \frac{127,979,099,307,513,182,492,153,794,673}{7,516,192,768,000,000,000,000,000,000}\right)$ = 17.02. Therefore $a + \frac{N - a^7}{7a^6}$ is (= 32000 + 17.02) = 32017.02; and consequently 32017.02 will be nearly equal to the seventh root of the proposed number 34,487,717,467,307,513,182,492,153,794,673. Q. E. I.

This number 32017.02 is exact in the six first figures 32017.0, and errs only in the seventh figure 2, which ought to be a cypher instead of a 2, because the true value of this seventh root is 32017.000,000,000, &c, *ad infinitum*, or the whole number 32017.

Art. 22. This expression $a + \frac{N - a^m}{m a^{m-1}}$, or $a + \frac{N - a^7}{7a^6}$, is

so

ſo much ſimpler than either of Mr. *de Lagny*'s expreſſions above-mentioned, and ſo much leſs difficult to be computed, that I am inclined to agree with Mr. *Raphſon* in thinking it, upon the whole, preferable to them. But, perhaps, when a, or the firſt value of the root ſought, conſiſts of only one figure, it may ſometimes be adviſeable to make uſe of one of Mr. *de Lagny*'s expreſſions, in order to obtain a ſecond near value of the root ſought, and then to make uſe of Mr. *Raphſon*'s expreſſion in order to obtain a third near value of it.

Art. 23. Theſe four examples are, I preſume, ſufficient to illuſtrate *Monſieur de Lagny*'s method of extracting the mth root of any propoſed number denoted by the letter N, by means of either of the two expreſſions $a + \frac{\overline{N - a^m} \times 2a}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ and $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{N - a^m}{m \times \frac{m-1}{2} \times a^{m-2}}}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} + \frac{2 \times \overline{N - a^m}}{m \times \overline{m-1} \times a^{m-2}}}$, when a, or the firſt near value of $\sqrt[m]{N}$, which is ſuppoſed to be already known, is leſs than its true value; which is the caſe ſuppoſed in the foregoing Problem. I ſhall therefore now proceed to conſider the other caſe, in which a, or the firſt near value of the mth root of the propoſed number N, is greater than its true value, and to inveſtigate ſimilar expreſſions for a ſecond near value of the ſaid root that ſhall approach nearer than a to its true value. This may be done by a ſolution of the following Problem.

PROBLEM

PROBLEM II.

Art. 24. Let N be any propofed number whatfoever, and m any propofed whole number whatfoever; and let a be a known number that is nearly equal to, but fomewhat greater than, the mth root of the given number N. It is required to find a fecond near value of the faid mth root of the given number N, that fhall approach much nearer to it than a, or the former near value of it that is already known.

SOLUTION.

Let z be put for the unknown difference between a, the firft near value of the mth root of the given number N, and the true value of the faid root. Then, fince a is fuppofed to be greater than $\sqrt[m]{N}$, and to exceed it by the difference z, it follows that $a - z$ will be $= \sqrt[m]{N}$, and confequently that $\overline{a - z}|^{m}$ will be $=$ N.

But, by Sir *Ifaac Newton*'s refidual theorem in the firft and fimpleft cafe of it, to wit, the cafe of integral powers, $\overline{a - z}|^{m}$ will be $=$ the feries $a^{m} - m a^{m-1} z + m \times \frac{m-1}{2} \times a^{m-2} z^{2} - m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} z^{3} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4} z^{4} - m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times a^{m-5} z^{5} +$ &c, continued to $m + 1$ terms; or, or, if, for the ſake of brevity, we ſubſtitute the capital letters A, B, C, D, E, F, &c, inſtead of the ſeveral numeral coefficients 1, m, $m \times \frac{m-1}{2}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, &c, reſpectively, $\overline{a-z}|^m$ will be = the ſeries $A a^m - B a^{m-1} z + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5$ + &c, continued to $m + 1$ terms. Therefore the ſaid ſeries $A a^m - B a^{m-1} z + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5$ + &c, continued to $m + 1$ terms, will be = N. This is the original equation, by the reſolution of which we are to find a near value of z, and conſequently of $a - z$, or a ſecond near value of $\sqrt[3]{N}$, which will approach nearer to it than a, or its former near value.

Art. 25. Now, ſince z is leſs, and uſually much leſs, than a, being about a 10th, or a 100th, part of it, or ſome ſtill leſſer part of it, it is evident that all the terms in the aforeſaid ſeries that involve zz, and z^3, and z^4, and the following powers of z, will be leſs, and uſually much leſs, than the term $B a^{m-1} z$, which involves only the ſimple power of z. And therefore, if all the ſaid terms of the ſeries be neglected or omitted, and the two firſt terms alone, to wit, the terms $A a^m - B a^{m-1} z$, be retained, the ſaid two terms alone will be nearly equal to the whole ſeries, and conſequently to the given number N; and therefore, if we add $B a^{m-1} z$ to both ſides, we ſhall have $A a^m$, nearly, = $N + B a^{m-1} z$, and (ſubtracting N from both ſides,) $A a^m - N$, nearly, = $B a^{m-1} z$, or $B a^{m-1} z$, nearly, = $A a^m - N$, and (dividing both ſides by $B a^{m-1}$,) we ſhall have

z, nearly, $= \frac{A a^m - N}{B a^{m-1}}$, or (becaufe A is $= 1$, and B is $= m$,) we fhall have z, nearly, $= \frac{a^m - N}{m a^{m-1}}$; which fraction may be derived from the known quantities N and a, by the common arithmetical operations of Multiplication, Subtraction, and Divifion. This therefore is an approximation to the true value of z, and confequently $a - \overline{\frac{a^m - N}{m a^{m-1}}}$ will be an approximation to the true value of $a - z$, or of $\sqrt[m]{N}$, or will be a fecond near value of it that will approach nearer to it than a, or the firft near value of it which was already known. And it will be ftill fomewhat greater (as the former value a was,) than the true value of $\sqrt[m]{N}$; as may be demonftrated in the manner following.

The whole feries $A a^m - B a^{m-1} z + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5 +$ &c, is $=$ N. Therefore (adding $B a^{m-1} z$ to both fides,) we fhall have the feries $A a^m + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5 +$ &c, $= N + B a^{m-1} z$, and (fubtracting N from both fides,) we fhall have the feries $A a^m - N + C a^{m-2} z^2 - D a^{m-3} z^3 + E a^{m-4} z^4 - F a^{m-5} z^5 +$ &c, $= B a^{m-1} z$. But, becaufe $C a^{m-2} z^2$ is greater than $D a^{m-3} z^3$, and $E a^{m-4} z^4$ is greater than $F a^{m-5} z^5$, and, in like manner, every following term in the faid feries that is marked with the fign $+$, is greater than the term immediately following it, which is marked with the fign $-$, it follows that the feries $A a^m - N + C a^{m-2} z^2 - D a^{m-3} z^3$

+

$+ Ea^{m-4}z^4 - Fa^{m-5}z^5 +$ &c, will be greater than its two firſt terms $Aa^m - N$. Therefore $Ba^{m-1}z$ (which is equal to the ſaid ſeries,) will be greater than $Aa^m - N$. Therefore z will be greater than $\frac{Aa^m - N}{Ba^{m-1}}$, and conſequently $a - z$, or $\sqrt[m]{N}$, will be leſs than $a - \frac{Aa^m - N}{Ba^{m-1}}$, or than $a - \frac{a^m - N}{ma^{m-1}}$, and therefore $a - \frac{a^m - N}{ma^{m-1}}$ will be greater than $a - z$, or the true value of the *m*th root of the propoſed number N. Q. E. D.

This quantity, $a - \frac{a^m - N}{ma^{m-1}}$, is the expreſſion given by Mr. Raphſon for the ſecond near value of the *m*th root of the given number N. And it is a very uſeful approximation: for it uſually gives us twice as many figures of the true value of $\sqrt[m]{N}$ exact as were exact in a, or the firſt near value of the ſaid *m*th root. And it is evidently the moſt ſimple and eaſy approximation to the value of the ſaid *m*th root that can well be imagined.

Art. 26. But Mr. de Lagny, being deſirous of finding at once a ſtill nearer value of the *m*th root of the number N, retains the third term $Ca^{m-2}z^2$, as well as the two firſt terms $Aa^m - Ba^{m-1}z$, of the ſeries $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2 - Da^{m-3}z^3 + Ea^{m-4}z^4 - Fa^{m-5}z^5 +$ &c, (which is equal to N,) and thereby converts the original equation $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2 - Da^{m-3}z^3 +$

$E a^{m-4} z^4 - F a^{m-5} z^5 + \&c, = N$ into a quadratick equation, to wit, the equation $A a^m - B a^{m-1} z + C a^{m-2} z^2 = N$, inſtead of converting it (as Mr. Raphſon does,) into the ſimple equation $A a^m - B a^{m-1} z = N$. And this quadratick equation he reſolves in two different ways, to wit, firſt, imperfectly, or inaccurately, and then accurately. By the former, or inaccurate, reſolution of it, he obtains a rational expreſſion for the value of z, and conſequently another rational expreſſion for the value of $a - z$, or for a ſecond near value of $\sqrt[3]{N}$, which is nearer to its true value than a, or the former near value of it, was; and by the accurate reſolution of the ſame quadratick equation he obtains a ſurd, or irrational, expreſſion for the value of z, and conſequently another ſurd, or irrational, expreſſion for the value of $a - z$, or for the ſecond near value of $\sqrt[3]{N}$, that approaches much nearer to its true value than its former near value, a, did. Theſe reſolutions of the ſaid quadratick equation $A a^m - B a^{m-1} z + C a^{m-2} zz = N$, may be performed in the following manner.

Art. 27. Since $A a^m - B a^{m-1} z + C a^{m-2} z^2$ is $= N$, we ſhall have $A a^m + C a^{m-2} z^2 = N + B a^{m-1} z$, and $A a^m = N + B a^{m-1} z - C a^{m-2} z^2$, and $A a^m - N = B a^{m-1} z - C a^{m-2} z^2$, or $B a^{m-1} z - C a^{m-2} z^2 = A a^m - N$; that is, $z \times B a^{m-1} - z \times C a^{m-2} z$ will be $= A a^m - N$, or $z \times \overline{B a^{m-1} - C a^{m-2} z}$ will be $= A a^m - N$. Therefore (dividing both ſides of the equation by the compound quantity $B a^{m-1} - C a^{m-2} z$,) we ſhall have $z = \frac{A a^m - N}{B a^{m-1} - C a^{m-2} z}$.

Now

Now let $\frac{Aa^m - N}{Ba^{m-1}}$, or $\frac{a^m - N}{ma^{m-1}}$, (which has already been ſhewn to be nearly equal to z,) be ſubſtituted inſtead of z in the ſecond term, $Ca^{m-2}z$, of the denominator of the fraction laſt obtained, to wit, the fraction $\frac{Aa^m - N}{Ba^{m-1} - Ca^{m-2}z}$.

And we ſhall have $z = \frac{Aa^m - N}{Ba^{m-1} - Ca^{m-2} \times \overline{\frac{a^m - N}{ma^{m-1}}}}$, or (becauſe A is = 1, and B is = m,)

$$z = \frac{a^m - N}{ma^{m-1} - Ca^{m-2} \times \overline{\frac{a^m - N}{ma^{m-1}}}}; \text{ which is}$$

$$(= \frac{a^m - N}{\frac{m^2a^{2m-2} - Ca^{m-2} \times a^m - N}{ma^{m-1}}}$$

$$= \frac{a^m - N}{\frac{m^2a^{2m-2} - Ca^{2m-2} + CNa^{m-2}}{ma^{m-1}}}$$

$$= \overline{a^m - N} \times \frac{ma^{m-1}}{m^2a^{2m-2} - Ca^{2m-2} + CNa^{m-2}}$$

$$= \overline{a^m - N} \times \frac{2aa \times ma^{m-1}}{2m^2a^{2m} - 2Ca^{2m} + 2CNa^m}$$

$$= \overline{a^m - N} \times \frac{2a \times ma^m}{2m^2a^{2m} - 2Ca^{2m} + 2CNa^m}$$

$$= \overline{a^m - N} \times \frac{2ma}{2m^2a^m - 2Ca^m + 2CN}$$

=

$$= \overline{a^m - N} \times \frac{2ma}{2m^2a^m - 2\times m\times\frac{m-1}{2}\times a^m + 2\times m\times\frac{m-1}{2}\times N}$$

$$= \overline{a^m - N} \times \frac{2ma}{2m^2a^m - m\times\overline{m-1}\times a^m + m\times\overline{m-1}\times N}$$

$$= \overline{a^m - N} \times \frac{2a}{2ma^m - \overline{m-1}\times a^m + \overline{m-1}\times N}$$

$$= \overline{a^m - N} \times \frac{2a}{2ma^m - ma^m + a^m + \overline{m-1}\times N}$$

$$= \overline{a^m - N} \times \frac{2a}{ma^m + a^m + \overline{m-1}\times N}$$

$$= \overline{a^m - N} \times \frac{2a}{\overline{m+1}\times a^m + \overline{m-1}\times N})$$

$= \frac{\overline{a^m - N}\times 2a}{\overline{m-1}\times N + \overline{m+1}\times a^m}$. Therefore z will be $=$ $\frac{\overline{a^m - N}\times 2a}{\overline{m-1}\times N + \overline{m+1}\times a^m}$, or $\frac{2a\times\overline{a^m - N}}{\overline{m-1}\times N + \overline{m+1}\times a^m}$; and confequently $a - z$ will be $=$

$a - \frac{2a\times\overline{a^m - N}}{\overline{m-1}\times N + \overline{m+1}\times a^m}$, or $a - \frac{2a\times\overline{a^m - N}}{\overline{m-1}\times N + \overline{m+1}\times a^m}$

will be a fecond near value of $\sqrt[m]{N}$, or of the mth root of the propofed number N. Q. E. I.

Art. 28. The accurate refolution of the quadratick equation $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2 = N$, may be performed as follows.

Since $Aa^m - Ba^{m-1}z + Ca^{m-2}z^2$ is $= N$, we fhall have $Aa^m + Ca^{m-2}z^2 = N + Ba^{m-1}z$, and $Aa^m = N + Ba^{m-1}z - C\ ^{m-2}z^2$, and $Aa^m - N$

=

$= Ba^{m-1}z - Ca^{m-2}z^2$, or $Ba^{m-1}z - Ca^{m-2}z^2 = Aa^m - N$, or (becauſe A is $= 1$, and B is $= m$, and C is $= m \times \frac{m-1}{2}$) $ma^{m-1}z - m \times \frac{m-1}{2}a^{m-2}z^2 = a^m - N$, and (multiplying both ſides by 2,) $2ma^{m-1}z - m \times \overline{m-1} \times a^{m-2}z^2 = 2 \times \overline{a^m - N}$, and, (dividing both ſides of the equation by $m \times \overline{m-1} \times a^{m-2}$,) $\frac{2ma^{m-1} \times z}{m \times \overline{m-1} \times a^{m-2}} - zz = \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}$, or $\frac{2a^{m-1} \times z}{\overline{m-1} \times a^{m-2}} - zz = \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}$, or (becauſe $\frac{a^{m-1}}{a^{m-2}}$ is $= a$,) $\frac{2a}{m-1} \times z - zz = \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}$.

Now $\frac{2a}{m-1} \times z - zz$ is $= z \times \overline{\frac{2a}{m-1} - z}$. And, by Euclid's Elements, Book II, Prop. 5, the rectangle or product, under z and $\frac{2a}{m-1} - z$, muſt be leſs than the ſquare of half the line $\frac{2a}{m-1}$. Therefore the compound quantity $\frac{2az}{m-1} - zz$ will be leſs than the ſquare of half $\frac{2a}{m-1}$, or than the ſquare of $\frac{a}{m-1}$, or than $\frac{aa}{\overline{m-1}|^2}$. Therefore $\frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}$, (which is equal to the compound quantity $\frac{2az}{m-1} - zz$,) will alſo be leſs than $\frac{aa}{\overline{m-1}|^2}$. Therefore both theſe quantities $\frac{2az}{m-1} - zz$ and $\frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}$ may be be ſubtracted from $\frac{aa}{\overline{m-1}|^2}$. Let them be ſo ſubtracted. And then we ſhall have $\frac{aa}{\overline{m-1}|^2} - \frac{2az}{m-1} + zz = \frac{aa}{\overline{m-1}|^2} - \frac{2 \times \overline{a^m - N}|}{m \times \overline{m-1} \times a^{m-2}}$. Therefore the ſquare-root of the trinomial quantity $\frac{aa}{\overline{m-1}|^2} - \frac{2az}{m-1} + zz$ will be equal to the ſquare root of the compound quantity $\frac{aa}{\overline{m-1}|^2} - \frac{2 \times \overline{a^m - N}|}{m \times \overline{m-1}| \times a^{m-2}}$; or, if, for the ſake of brevity, we denote the ſaid compound quantity by the capital letter P, the ſquare-root of the trinomial quantity $\frac{aa}{\overline{m-1}|^2} - \frac{2az}{m-1} + zz$ will be $= \sqrt{P}$. But, whenever z is leſs than $\frac{a}{m-1}$, (as is commonly the caſe in theſe extractions of the roots of numbers,) $\frac{a}{m-1} - z$ will be the ſquare-root of the trinomial quantity $\frac{aa}{\overline{m-1}|^2} - \frac{2az}{m-1} + zz$. Therefore $\frac{a}{m-1} - z$ will $= \sqrt{P}$, and conſequently $\frac{a}{m-1}$ will be $= \sqrt{P} + z$, and z will be $= \frac{a}{m-1} - \sqrt{P}$. Therefore $a - z$ will be ($= a - \overline{\frac{a}{m-1} - \sqrt{P}}) = a - \frac{a}{m-1} + \sqrt{P}$; and conſequently $a - \frac{a}{m-1} + \sqrt{P}$, or $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$ will be a ſecond near value of $\sqrt[m]{N}$, or of the mth root of the given number N. Q. E. I.

Art. 29. When m is $= 3$, the former of theſe two expreſſions,

preſſions, to wit, the rational expreſſion $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, will be $(= a - \frac{2a \times \overline{a^3 - N}}{2N + 4a^3}) = a - \frac{a \times \overline{a^3 - N}}{N + 2a^3}$; and the latter, or irrational, expreſſion, $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$ will be $(= a - \frac{a}{2} + \sqrt{\frac{aa}{4} - \frac{2 \times \overline{a^3 - N}}{3 \times 2 \times a}} = \frac{a}{2} + \sqrt{\frac{aa}{4} - \frac{a^3 - N}{3a}} = \frac{a}{2} + \sqrt{\frac{3a^3}{12a} - \frac{4a^3 - 4N}{12a}} = \frac{a}{2} + \sqrt{\frac{3a^3 - 4a^3 + 4N}{12a}}) = \frac{a}{2} + \sqrt{\frac{4N - a^3}{12a}}$. Therefore the two expreſſions for the ſecond near value of the cube-root of a given number N, when a, or the former near value of it, is greater than its true value, are $a - \frac{a \times \overline{a^3 - N}}{N + 2a^3}$ and $\frac{a}{2} + \sqrt{\frac{4N - a^3}{12a}}$.

Art. 30. And, when m is $= 5$, the former of the two foregoing general expreſſions, to wit, the rational expreſſion $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, will be $(= a - \frac{2a \times \overline{a^5 - N}}{4N + 6a^5})$ $= a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$; and the latter, or irrational, expreſſion, $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$ will be $(= a - \frac{a}{4} + \sqrt{\frac{aa}{16} - \frac{2 \times \overline{a^5 - N}}{5 \times 4 \times a^3}} = \frac{3a}{4} + \sqrt{\frac{aa}{16} - \frac{a^5 - N}{10a^3}}$ $= \frac{3a}{4} + \sqrt{\frac{10a^5}{160a^3} - \frac{16a^5 - 16N}{160a^3}} = \frac{3a}{4} + \sqrt{\frac{10a^5 - \overline{16a^5 - 16N}}{160a^3}}$

$= \frac{3a}{4} + \sqrt{\left(\frac{10a^5 - 16a^5 + 16N}{160a^3}\right)} = \frac{3a}{4} + \sqrt{\left(\frac{16N - 6a^5}{160a^3}\right)} = \frac{3a}{4} + \sqrt{\left(\frac{8N - 3a^5}{80a^3}\right)}$. Therefore the two expreſſions for the ſecond near value of the fifth root of a given number N, when a, or the former near value of the ſaid root, is greater than its true value, are $a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$ and $\frac{3a}{4} + \sqrt{\left(\frac{8N - 3a^5}{80a^3}\right)}$.

Art. 31. And, when m is $= 7$, the former of the two foregoing general expreſſions, to wit, the rational expreſſion $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$, will be $(= a - \frac{2a \times \overline{a^7 - N}}{6N + 8a^7}) = a - \frac{a \times \overline{a^7 - N}}{3N + 4a^7}$; and the latter, or irrational, expreſſion, $a - \frac{a}{m-1} + \sqrt{\left(\frac{aa}{\overline{m-1}|^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}\right)}$ will be $(= a - \frac{a}{6} + \sqrt{\left(\frac{aa}{36} - \frac{2 \times \overline{a^7 - N}}{7 \times 6 \times a^5}\right)} = \frac{5a}{6} + \sqrt{\left(\frac{aa}{36} - \frac{a^7 - N}{21a^5}\right)}$ $= \frac{5a}{6} + \sqrt{\left(\frac{21a^7}{36 \times 21a^5} - \frac{36a^7 - 36N}{36 \times 21a^5}\right)} = \frac{5a}{6} + \sqrt{\left(\frac{21a^7 - \overline{36a^7 - 36N}}{36 \times 21a^5}\right)}$ $= \frac{5a}{6} + \sqrt{\left(\frac{21a^7 - 36a^7 + 36N}{36 \times 21a^5}\right)} = \frac{5a}{6} + \sqrt{\left(\frac{36N - 15a^7}{36 \times 21a^5}\right)} = \frac{5a}{6} + \sqrt{\left(\frac{12N - 5a^7}{36 \times 7a^5}\right)}) = \frac{5a}{6} + \sqrt{\left(\frac{12N - 5a^7}{252a^5}\right)}$. Therefore the two expreſſions for the ſecond near value of the 7th root of a given number N, when a, or the former near value of the ſaid root, is greater than its true value, are $a - \frac{a \times \overline{a^7 - N}}{3N + 4a^7}$ and $\frac{5a}{6} + \sqrt{\left(\frac{12N - 5a^7}{252a^5}\right)}$.

Examples of the Extraction of the Roots of given Numbers by means of the Two General Expressions $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ *and* $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}|^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$, *which have been found by the Solution of Problem II, when* a, *or the First near Value of* $\sqrt[m]{N}$, *is greater than its true Value.*

EXAMPLE I.

Art. 32. Let it be required to find the fifth root of 2, which has been already investigated by means of the two expressions investigated in Problem I, and found to be = 1.148,697,34. And let us suppose that we have already discovered that this root is greater than 1.14, but less than 1.15, and differs less from 1.15 than from 1.14; so that 1,15 may be taken for a, or its first near value. Then, by art. 30, the two expressions of the second near value of this root will be $a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$ and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$, in which 2 must be substituted instead of N, and 1.15 instead of a.

Now, since N is = 2, and a is = 1.15, we shall have 2N (= 2 × 2) = 4, and a^3 = 1.520,875, and a^5 = 2.011,357,187,5, and a^5 — N (= 2.011,357,187,5 — 2.000,000,000,0) = 0.011,357,187,5, and $a \times \overline{a^5 - N}$ (= 1.15 × 0.011,357,187,5) = 0.013,060,765,625, and $3a^5$ (= 3 × 2.011,357,187,5) = 6.034,071,562,5, and 2N + $3a^5$ (= 4 + 6.034,071,562,5) = 10.034,071,562,5, and $\frac{a \times \overline{a^5 - N}}{2N + 3a^5}$ (= $\frac{0.013,060,765,625}{10.034,071,562,5}$) = 0.001,301,641, and

 conse-

consequently $a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$ (= 1.150,000,000 − 0.001, 301,641) = 1.148,698,359. Therefore 1.148,698,359 will be a second near value of the fifth root of the given number 2. Q. E. I.

And, since a is = 1.15, and a^3 is = 1.520,875, and a^5 is = 2.011,357,187,5, and $3a^5$ is = 6.034,071,562,5, we shall have $80a^3$ (= 80 × 1.520,875) = 121.670,000, and $8N - 3a^5$ (= 8 × 2 − 6.034,071,562,5 = 16.000,000, 000,0 − 6.034,071,562,5,) = 9.965,928,437,5, and $\frac{8N - 3a^5}{80a^3}$ $\left(= \frac{9.965,928,437,5}{121.670,000}\right)$ = 0.081909,496486, and $\sqrt{\frac{8N - 3a^5}{80a^3}}$ (= $\sqrt{}$ 0.081909,496486,) = 0.286,198,351, and $3a$ (= 3 × 1.15) = 3.45, and $\frac{3a}{4}$ $\left(= \frac{3.45}{4}\right)$ = 0.8625, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$ (= 0.8625 + 0.286,198,351) = 1.148,698,351. Therefore 1.148,698,351 will be a second near value of the fifth root of the given number 2.

Q. E. I.

As these two approximations to the fifth root of 2, to wit, 1.148,698,359 and 1.148,698,351, agree with each other in the first nine figures 1.148,698,35, we may reasonably conclude that those nine figures are exact, or are the first nine figures of a more accurate value of the said fifth root.

Art. 33. If we make use of Mr. Raphson's expression, to wit, $a - \overline{\frac{a^m - N}{ma^{m-1}}}$, or $a - \overline{\frac{a^5 - N}{5a^4}}$, for the purpose of obtaining a second near value of the fifth root of 2, after 1.15 has been taken for a, or, its first near value, the computation of it will be as follows.

Since a is = 1.15, we shall have a^4 = 1.749,006,25, and a^5 = 2.011,357,187,5, and $5a^4$ (= 5 × 1.749,006,25) =

$= 8.745,031,25$, and $a^5 - N$ ($= 2.011,357,187,5 - 2$) $= 0.011,357,187,5$, and $\frac{a^5 - N}{5a^4}$ ($= \frac{0.011,357,187,5}{8.745,031,25}$) $= 0.001,298,7$, and $a - \overline{\frac{a^5 - N}{5a^4}}$ ($= 1.150,000,0 - 0.001,298,7$) $= 1.148,701,3$. Therefore 1.148,701,3 will be the ſecond near value of the fifth root of 2, reſulting from Mr. Raphſon's expreſſion $a - \overline{\frac{a^5 - N}{5a^4}}$. Q. E. I.

This number 1.148,701,3, is greater than the true value of the 5th root of 2, to wit, the number 1.148,698,3, but exceeds it by only the very ſmall quantity 0.000,003,3.

EXAMPLE II.

Art. 34. Let it be required to find the 5th root of the number 2,327,834,559,873.

Now this number, which conſiſts of thirteen figures, is greater than 100,000, or the fifth power of 10; and it is likewiſe greater than 10,000,000,000, or the fifth power of 100: but it is leſs than 1,000,000,000,000,000, or the fifth power of 1000. Therefore its fifth root muſt be greater than 100, but leſs than 1000.

Further, this number, 2,327,834,559,873, is greater than 2,320,000,000,000, or than 232 × 10,000,000,000, or than 232 × the fifth power of 100. Therefore the fifth root of 2,327,834,559,873 will be greater than 100 × the fifth root of 232. But the difference will not be great; and conſequently, if we can find a number that ſhall be nearly equal to the fifth root of 232, we need only multiply the ſaid number into 100, and the product will be nearly equal to the fifth root of 2,320,000,000,000, and therefore will likewiſe

likewise be pretty nearly equal to the fifth root of the proposed number 2,327,834,559,873, so as to be a convenient first near value of the said fifth root, and a proper basis to found a further approximation upon to a second near value of the said fifth root, by either of the two foregoing expressions of Mr. de Lagny, which have been investigated above in the Solution of Problem II, or by Mr. Raphson's expression. We will therefore endeavour to find the fifth root of the number 232.

Art. 35. Now the fifth power of the number 2 is 32, which is much less than 232; and the fifth power of 3 is 243, which is a little greater than 232. Therefore the fifth root of 232 must be much greater than 2, and a little less than 3. We may therefore reasonably conjecture that it will be nearly equal to $2 + \frac{9}{10}$, or 2.9. We will therefore suppose it to be $= 2.9$, and try what the result of that supposition will be.

Now the fifth power of 2.9 is 205.11149, which is less than 232. Therefore the fifth root of 232 will be greater than 2.9. And, as 232 differs much less from 243, or the the fifth power of 3, than from 205.11149, or the fifth power of 2.9, we may reasonably suppose that the fifth root of 232 will differ much less from 3 than from 2.9; and therefore we will suppose that it is nearly $= 2.98$, and will raise the said number 2.98 to its fifth power, in order to examine the truth of the said supposition.

Now $\overline{2.98}|^5$ is $=$ 235.007,282,396,8; which is nearly equal to, but a little greater than, 232. Therefore 2.98 must be nearly equal to, but a little greater than, the fifth root of 232; and consequently 100 × 2.98 must be nearly equal to, but a little greater than, the fifth root of 232 × 10,000,000,000; or 298 must be nearly equal to, but a little greater than, the fifth root of 2,320,000,000,000. And, further, since 2.98 is the fifth root of the number 235.007,282,396,8, the number 100 × 2.98, will be the fifth root of the number 235.007,282,396,8 × 10,000,000, 000;

000; that is, the number 298 will be the fifth root of the number 2,350,072,823,968, which is greater than the proposed number 2,327,834,559,873. Therefore 298 will be greater than the fifth root of the said number 2,327,834,559,873. But it will be near enough to it to make it a very convenient basis of a further approximation to the true value of the fifth root of the said number, 2,327,834,559,873, by means of the two expressions $a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$, and $\frac{3a}{4} + \sqrt{\frac{8N - 3a^5}{80a^3}}$.

Art. 36. Here then we shall have N = 2,327,834,559,873, and $a = 298$. Therefore a^3 will be = 26,463,592, and a^4 will be = 7,886,150,416, and a^5 will be = 2,350,072, 823,968; and consequently a^5 — N will be (= 2,350,072, 823,968 — 2,327,834,559,873) = 22,238,264,095, and $\overline{a^5 - N} \times a$ will be (= 22,238,264,095 × 298) = 6,627,002,700,310, and 2N will be (= 2 × 2,327,834, 559,873) = 4,655,669,119,746, and $3a^5$ will be (= 3 × 2,350,072,823,968) = 7,050,218,471,904, and 2N + $3a^5$ will be (= 4,655,669,119,746 + 7,050,218,471,904) = 11,705,887,591,650, and $\frac{\overline{a^5 - N} \times a}{2N + 3a^5}$ will be ($= \frac{6,627,002,700,310}{11,705,887,591,650}$) = 0.566,125,605, and consequently $a - \frac{a \times \overline{a^5 - N}}{2N + 3a^5}$ will be (= 298.000,000,000 − 0.566,125, 605) = 297.433,874,395. Therefore 297.433,874,395 will be a second near value of the fifth root of the proposed number 2,327,834,559,873. Q. E. I.

And $3a$ will be (= 3 × 298) = 894, and $\frac{3a}{4}$ will be ($= \frac{894}{4}$) = 223.5, and a^3 will be ($= \overline{298}|^3$) = 26,463,592, and 8N will be (= 8 × 2,327,834,559,873) = 18,622, 676,478,984, and a^5 will be ($= \overline{298}|^5$) = 2,350,072, 823,968, and $3a^5$ will be (= 3 × 2,350,072,823,968) = 7,050,218,471,904, and 8N — $3a^5$ will be (= 18,622, 676, 676,478,984 — 7,050,218,471,904) = 11,572,458,007,080, and $80a^3$ will be (= 80 × 26,463,592) = 2,117,087,360, and $\frac{8N - 3a^5}{80a^3}$ will be (= $\frac{11,572,458,007,080}{2,117,087,360}$) = 5466.216, 569,863,2, and $\sqrt{\frac{8N - 3a^5}{80a^3}}$ will be (= $\sqrt{5466.216569,8632}$) = 72.933,866,190, and consequently $\frac{3a}{4}$ + $\sqrt{\frac{8N - 3a^5}{80a^3}}$ will be (= 223.5 + 72.933,866,190) = 297.433,866,190. Therefore 297.433,866,190 will be a second near value of the fifth root of the proposed number 2,327,834,559,873. Q. E. I.

These two numbers 297.433,874,395, and 297.433,866, 190, agree with each other in the first seven figures 297. 433,8. Therefore we may conclude that these seven figures are exact, or are the first seven figures of a more accurate value of the fifth root of the proposed number 2,327,834, 559,873.

Art. 37. If we make use of Mr. Raphson's expression, to wit, $a - \frac{a^m - N}{ma^{m-1}}$, or $a - \frac{a^5 - N}{5a^4}$, for the purpose of obtaining a second near value of the fifth root of the proposed number 2,327,834,559,873, after 298 has been taken for a, or its first near value, which is somewhat greater than the truth, the computation of it will be as follows.

Since a is = 298, we shall have a^4 = 7,886,150,416, and a^5 = 2,350,072,823,968, and a^5 — N (= 2,350,072, 823,968 — 2,327,834,559,873) = 22,238,264,095, and $5a^4$ (= 5 × 7,886,150,416) = 39,430,752,080, and $\frac{a^5 - N}{5a^4}$ (= $\frac{22,238,264,095}{39,430,752,080}$) = 0.563,982, and $a - \frac{a^5 - N}{5a^4}$ (= 298.000,000 — 0.563,982) = 297.436,018. Therefore 297.436,018 will be a second near value of the fifth root of the proposed number 2,327,834,559,873. Q. E. I.

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The firſt five figures, 297.43, of this number, 297.436,018, obtained by Mr. Raphſon's expreſſion, are exact.

And, if we make $a = 297.436$, and repeat the application of Mr. Raphſon's expreſſion, we ſhall obtain the value of the fifth root of the ſaid propoſed number 2,327,834, 559,873, to a much greater degree of exactneſs. This may be done in the manner following.

If a is taken $=$ 297.436, we ſhall have

$a^4 = 7,826,617,827.880,165,417,216$, and

$a^5 = 2,327,917,900,253.364,881,035,058,176$, and

$a^5 - N$ $(= 2,327,917,900,253.364,881,035,058,176$
$- 2,327,834,559,873.000,000,000,000,000)$
$= 83,340,380.364,881,035,058,176$,

and $5a^4$ $(= 5 \times 7,826,617,827.880,165,417,216) =$ 39,133,089,139.400,827,086,080, and conſequently $\frac{a^3 - N}{5a^4}$ $\left(= \frac{83,340,380.364,881,035,058,176}{39,133,089,139.400,827,086,080}\right) = 0.002,129,662$, and $a - \frac{a^3 - N}{5a^4}$ $(= 297.436,000,000 - 0.002,129,662) =$ 297 433,870,338. Therefore 297.433,870,338 will be the more accurate value of the fifth root of the propoſed number 2,327,834,559,873. Q. E. I.

Of this number, 297.433,870,338, which we have now found for the laſt near value of the fifth root of 2,327,834, 559,873, I believe the firſt ten figures 297.453,870,3 to be exact, if no miſtakes have been made in the calculation. Yet Mr *Raphſon* (from whoſe *Analyſis Æquationum Univerſalis*, Problem IV, page 12, this example is taken,) makes this fifth root equal to 297.433,874,895. But I believe the four laſt figures, 4895, of this number to be erroneous; becauſe in Mr. *Raphſon*'s laſt proceſs, (of which this number, 297.433,874,895, is the reſult,) the value of g (which anſwers to a in our notation,) was taken equal only to 297.46, which is exact only in the firſt four figures 297.4, whereas in the laſt proceſs of the foregoing computation we took a equal to 297.436, which is exact in the firft five figures, 297.43; and confequently the number refulting from this fuppofition ought to be more exact than that which refults from the other lefs accurate fuppofition made by Mr. *Raphfon.* But we may, at leaft, conclude that the firft eight figures, 297.433,87, of thofe two numbers, which are the fame in both, are exact, or are the firft eight figures of a number approaching more nearly than either of them to the true value of the fifth root of the propofed number 2,327,834,559,873.

Art. 38. Thefe two examples will, I prefume, be fufficient to illuftrate *Monfieur de Lagny*'s method of extracting the mth root of any propofed number N, by means of either of the two expreffions, $a - \frac{2a \times \overline{a^m - N}}{\overline{m-1} \times N + \overline{m+1} \times a^m}$ and $a - \frac{a}{m-1} + \sqrt{\frac{aa}{\overline{m-1}^2} - \frac{2 \times \overline{a^m - N}}{m \times \overline{m-1} \times a^{m-2}}}$, when a, or the firft near value of $\sqrt[m]{N}$, which is fuppofed to be already known, is greater than its true value; which is the cafe fuppofed above in Problem II, by the folution of which thofe expreffions were obtained. I have therefore nothing more to add concerning the explanation of *Monfieur de Lagny*'s method aforefaid. But I will juft make another obfervation, or two, concerning the faid method of extracting the roots of numbers, as compared with other methods of performing the fame thing.

Observations on the several different Methods that may be taken for Extracting the Roots of Numbers.

Art. 39. In the 1st place, then, it is manifest that either of Mr. *de Lagny*'s two expressions, the rational one and the irrational one, for obtaining a second near value of the root of a given number, when a former near value of it is already known, is greatly to be preferred to *the common*, or, rather, *the old*, method of extracting such root, by which, with a great deal of trouble, we obtain only one new figure of the root sought by every new process; except, perhaps, in extracting the square-root of a number, which is easy enough in the common way, (at least for the first three or four figures of the root sought,) to make it unnecessary to have recourse to other methods. But in extracting the cube-root, or the fifth root, or the seventh root, or any higher root, of a proposed number, the case is very different, and it will be found highly expedient to have recourse either to Mr. *de Lagny*'s method of extracting them, or to Mr. *Raphson*'s, or to some other method of performing the said extraction.

Secondly, if the *m*th root of any number is to be found only to four, or five, figures, it will be most adviseable to have recourse to a Table of Logarithms for this purpose. For, by the use of such a Table, we may always obtain any proposed root of a given number exact to four, or five, places of figures, with very great ease, and without making use of the proportional parts set down in those tables, and which are necessary to the obtaining the said roots exactly to six, or seven, or more, places of figures. Whenever therefore we want to find the *m*th root of a proposed number only to four, or five, places of figures, it seems best to have recourse at once to a Table of Logarithms for that purpose.

But, 3dly, if we wish to obtain the *m*th root of any number, exactly to nine, or ten, or more, places of figures, it will be convenient to have recourſe to either Mr. *de Lagny*'s or Mr. *Raphſon*'s methods of approximation for that purpoſe. And, if we wiſh to obtain the ſaid root exact only to nine places of figures, I ſhould think it would be expedient to make uſe of Mr. *Raphſon*'s expreſſion for that purpoſe, in preference to either of Mr. *de Lagny*'s expreſſions, as being ſimpler and eaſier to compute than the latter; but, if we wiſh to obtain the ſaid root exact to fourteen, or fifteen, places of figures, I ſhould think it would be moſt adviſeable to have recourſe to one of Mr. *de Lagny*'s expreſſions for that purpoſe, rather than to make uſe of Mr. *Raphſon*'s expreſſion, and repeat the proceſs a ſecond time, as was done above in art. 37. And of Mr. *de Lagny*'s two expreſſions, the latter, or irrational, expreſſion will be found leſs troubleſome to compute, and, uſually, in a ſmall degree more exact, than the rational expreſſion. But it may often be prudent to compute them both, to be checks upon each other; and the number of figures in which the reſults of both expreſſions are found to agree, may be juſtly concluded to be exact.

And, 4thly, when we make uſe of either Mr. *Raphſon*'s or Mr. *de Lagny*'s methods of extracting the *m*th root of a given number, I conceive it will be always adviſeable to make uſe of a Table of Logarithms firſt, in order to obtain the firſt near value of the root ſought, from which we are afterwards to derive a ſecond and more accurate near value of it, by means of the expreſſions invented by thoſe ingenious Gentlemen. This, indeed, is not abſolutely neceſſary, as it is always eaſy to find the propoſed root exact to one, or two, figures, by ſome very ſimple reaſonings and trials, as is ſhewn above in all the foregoing examples. But it will always be ſtill eaſier to find theſe firſt figures by the help of a Table of Logarithms, and we may find them by that means not only to two places of figures, but to five.

End of the Tract, intitled, Mr. de Lagny'*s General Method of Extracting the Roots of Numbers by Approximation.*

OBSER-

OBSERVATIONS

ON

MR. RAPHSON'S METHOD

OF

RESOLVING AFFECTED EQUATIONS OF ALL DEGREES

BY

APPROXIMATION.

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MR. RAPHSON'S METHOD

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RESOLVING AFFECTED EQUATIONS OF ALL DEGREES

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APPROXIMATION.

Article 1. IN the foregoing Tract I have given a pretty full explanation of *Monsieur de Lagny*'s Method of Extracting the Roots of Numbers by Approximation, and I have likewise mentioned Mr. *Raphson*'s more simple and easy, though less exact, method of performing the same thing. But both these methods may be applied to the resolution of all sorts of equations, those which are called *affected* equations *, or in which the unknown quantity occurs in

* This expression of *affected equations* seems to require some further explanation. It was introduced by the celebrated Vieta, the great father and restorer of Algebra. He has many expressions peculiar to himself, and which have not been adopted by subsequent Algebräists. Amongst these are the following ones. He calls a set of quantities in continual geometrical proportion, (such as the quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c,) a set of *scalar* quantities, or *magnitudines scalares*; and, when there are several of these *scalar* quantities connected with each other by the signs + and −, or by Addition and Subtraction, (as in the compound quantity $x^5 + ax^4 - b^2x^3$,) he calls the highest quantity, or that which is farthest in the scale of quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c, (to wit, the quantity x^5 in the said compound quantity $x^5 + ax^4 - b^2x^3$,) *the power* of the fundamental quantity x, or of the second term in the said scale; and he calls the lower scalar quantities, which are involved in the second and third terms of the said compound quantity $x^5 + ax^4 - b^2x^3$, to wit, the quantities

in more than one term, as well as those which are called *pure* equations, or in which the unknown quantity occurs in only one term, and which are resolved by the mere extraction of the roots of given numbers. And in all affected equations beyond biquadraticks, or those of the fourth power, these methods of approximation are the only methods that can be taken for discovering their roots, or the values of the unknown quantities contained in them. And even in cubick and biquadratick equations, though particular methods have been invented by Mathematicians, for the accurate resolution of most of the cases of these equations, (to wit, the rules called *Cardan's rules* for the resolution of most cases of cubick equations, and the rules invented by *Lewis Ferrari* of *Bologna* in Italy, about the year 1545, and explained at large in *Bombelli's* Algebra, in the year 1579, and those afterwards invented by *Monsieur Des Cartes*, and published in his Geometry in the year 1637, for

ties x^4 and x^3, (or, in our present language, the inferiour powers of x,) scalar quantities of a *parodic* degree to x^5, or *the power* of the fundamental quantity x. This word *parodic* I take to be derived (though Vieta does not tell us so,) from the Greek words παρὰ and ὁδὸς, which signify *near* and *a way*, or *road*, because these inferiour scalar quantities, x^3 and x^4, lie *in the way* as you pass along in the scale of the aforesaid quantities 1, x, x^2, x^3, x^4, x^5, x^6, x^7, &c, from 1 to x^5, which he calls *the power* of x in the said compound quantity $x^5 + ax^4 - b^2x^3$. These inferiour scalar quantities x^3 and x^4 are therefore *parodic*, or *situated in the way to*, or are *leading to*, the said *power*, or higher scalar quantity, x^5. He then proceeds to define *a pure power* and *an affected power*, and tells us, that *a pure power* is a scalar quantity that is not affected with, or mixed with, any *parodic*, or *inferiour* scalar quantity, and that *an affected power* is a scalar quantity that is mixed, or connected by Addition, or Subtraction, with one, or more, *inferiour*, or *parodic*, scalar quantities, combined with co-efficients that raise them to the same dimension as the *power* itself, or make them *homogeneous* to it, and consequently capable of being added to it, or subtracted from it. Thus x^5 alone is *a pure power* of x, namely, its fifth power; and $x^5 + ax^4 - b^2x^3$ is *an affected power* of x, namely, its fifth power *affected by*, or *connected with*, the two *parodic*, or *inferiour* scalar quantities, x^3 and x^4, which are multiplied into bb and a, in order to make them *homogeneous to*, or *of the same dimension with*, x^5 itself, and consequently capable of being added to it, or subtracted from it. See Schooten's edition of Vieta's Works, published at Leyden in Holland, in the year 1646, pages 3 and 4.

This, then, being the meaning of the expressions *a pure power* and *an affected power*, the meaning of the corresponding expressions of *a pure equation* for the refolution of biquadratick equations, by the mediation of cubick equations,) it will be found that thefe methods of approximation will, for the moft part, enable us to find the values of their roots to any propofed degree of exactnefs, with lefs trouble than the particular and accurate methods above-mentioned, which have been invented for that purpofe. So that thefe methods of refolving equations by approximation ought to be confidered as of the higheft utility, and as being abfolutely neceffary to the completion of the Doctrine of the Refolution of Algebräick Equations, which is the moft important branch of the Science of Algebra.

Art. 2. But it is not fo eafy to determine, which of thefe two methods of approximation, Mr. *Raphfon*'s, or Mr. *de Lagny*'s, deferves to be preferred to the other on thefe occafions. Mr. *Raphfon*'s is certainly much fimpler than the other, becaufe it proceeds by confidering the new, or tranfformed, equation, (refulting from the fubftitution of $a + z$,

tion and *an affected equation* follows from it of courfe: *a pure equation* fignifying an equation in which a pure power of an unknown quantity is declared to be equal to fome known quantity; fuch as the equation $x^5 = 79$; and *an affected equation* fignifying an equation in which a power of an unknown quantity affected by, or connected, either by Addition or Subtraction, with, fome inferiour powers of the fame unknown quantity, (multiplied into proper co-efficients in order to make them *homogeneous* to the faid higheft power of the faid unknown quantity,) is declared to be equal to fome known quantity; fuch as the equation $x^5 + ax^4 - b^2x^3 = 79$. This I take to be the original meaning of the expreffion *an affected equation*. But, as the language of *Victa* has not been adopted by fubfequent writers of Algebra, I fhould think it would be more convenient to call them by fome other name. And, perhaps, thofe of *binomial*, *trinomial*, *quadrinomial*, *quinquinomial*, and, in general, that of *multinomial* equations, would be as convenient as any. Thus, $xx + ax = rr$, and $x^3 + ax^2 = r^3$, and $x^3 + a^2x = r^3$, and $x^4 + a^3x = r^4$, and $x^4 + ax^3 = r^4$, might all be called *binomial* equations, becaufe they would be equations in which a *binomial* quantity, or quantity confifting of two terms that involved the unknown quantity x, is declared to be equal to a known quantity; and, for a like reafon, the equations $x^3 + ax^2 + b^2x = r^3$, and $x^4 - ax^3 + b^2x^2 = r^4$, and $x^4 - ax^3 + b^3x = r^4$, and $x^5 + ax^4 + b^2x^3 = r^5$, and $x^5 + ax^4 - b^2x^3 = r^5$, and $x^5 + b^2x^3 + c^4x = r^5$, might be called *trinomial* equations. And the like names might be given to equations of a greater number of terms. Dr. Hutton, I obferve, in his excellent new Mathematical and Philofophical Dictionary, juft now publifhed, (Feb. 2, 1795,) calls them *compound* equations; which is likewife a very proper name for them, and lefs obfcure than that of *affected* equations.

 or

or $a - z$, inſtead of x, in the original equation,) as being only a ſimple equation, and reſolving it accordingly, or by the mere operation of Diviſion; whereas, in Mr. *de Lagny*'s method, the ſaid new, or transformed, equation is conſidered as a quadratick equation, and reſolved accordingly; which, when a (or the firſt near value of the root, that is ſuppoſed to be already known,) is a number conſiſting of five, or ſix, figures, produces a great deal of labour, and often a great deal of perplexity. I am therefore inclined to give the preference to Mr. *Raphſon*'s method in reſolving all affected equations, more eſpecially when the number a conſiſts of more than two figures: but it muſt be confeſſed that the celebrated Dr. *Halley* (who had much experience, and was an excellent judge of theſe matters,) was of a different opinion, and gave the preference to Mr. *de Lagny*'s method, which he has therefore taken the pains to explain in a better manner than had been done by *Monſieur de Lagny* himſelf, and likewiſe to illuſtrate by examples, in his Tract in the Philoſophical Tranſactions, Number 210, intitled, "*A New, Exact, and Eaſy Method, of finding the* "*Roots of any Equations Generally, and that without any pre-* "*vious Reduction,*" which was publiſhed in the year 1694. On the other hand we may obſerve, that Mr. *Raphſon* always continued to give his own method the preference, after the publication of the tracts of *Monſieur de Lagny* and Dr. *Halley* upon the ſubject, as well as before their publication, when he tells us he had himſelf had the thought of adopting the principle which was afterwards followed by Mr. *de Lagny* and Dr. *Halley*, of treating the transformed equation as a quadratick equation, but had deliberately rejected it on account of the greater eaſe and ſimplicity of the other method, in which the ſaid transformed equation is conſidered and treated as a ſimple equation. And Sir *Iſaac Newton* in his method of reſolving equations by approximation (which differs very little from Mr. *Raphſon*'s,) ſeems alſo to prefer Mr. *Raphſon*'s practice, of treating the transformed equation as a mere ſimple equation, to that of Mr. *de Lagny* and Dr. *Halley*, of treating the ſaid equation as a quadratick equation. I therefore cannot but recommend it to all young Algebräiſts to ſtudy Mr. *Raphſon*'s excellent Trea-

tiſe

tiſe on this ſubject, intitled, *Analyſis Æquationum Univerſalis*, with great attention, and to endeavour to make themſelves maſters of it, by going carefully through all the examples given in it, and performing all the arithmetical operations contained in them. And I will venture to ſay that they will thereby acquire more uſeful knowledge in Algebra, towards the buſineſs of reſolving affected, or compound, or multinomial, equations, than by reading all that has been written by *Harriot* and *Des Cartes*, and his learned Commentator *Van Schooten*, and all his other Commentators, and their numerous followers, on the boaſted doctrine of the Generation of Equations one from another, by ſuppoſing $x - a$ to be $= 0$, and $x - b$ to be $= 0$, and $x - c$ to be $= 0$, and $x + d$ to be $= 0$, and $x + e$ to be $= 0$, and ſo on; and then multiplying the binomial quantities $x - a$, $x - b$, $x - c$, $x + d$, $x + e$, &c, into each other, and likewiſe all the abſtruſe and intricate matter that has been delivered by Sir *Iſaac Newton*, and Mr. *Graveſende* and Mr. *Mac Laurin*, and other learned Algebräiſts of modern times, on the invention of Diviſors, which is grounded on that doctrine of the Generation of Equations from each other.

Art. 3. Yet in reading this excellent Treatiſe of Mr. *Raphſon*, which I ſo much recommend, there will now and then occur ſome difficulties which are not inherent in the ſubject itſelf, but which might have been avoided, if Mr. *Raphſon* had not unfortunately adopted the perplexing doctrines of modern writers of Algebra, about negative quantities and negative roots of Equations. The quantities called *negative* are ſuch as it is impoſſible to form any clear idea of, being defined, by Sir *Iſaac Newton* and other Algebräiſts *,

* Quantitates vel *Affirmativæ* ſunt, ſeu majores Nihilo, vel *Negativæ*, ſeu Nihilo minores.——Newton's *Arithmetica Univerſalis*, page 3.

When a greater quantity is taken from a leſſer of the ſame kind, the remainder becomes of the oppoſite kind.——*Mac Laurin*'s *Algebra*, page 5.

An affirmative quantity is a quantity greater than nothing, and is known by this ſign, +; a negative quantity is a quantity leſs than nothing, and is known by this ſign, −.——*Saunderſon*'s *Algebra*, Vol. I. page 50, article 2.

to be such quantities as are *less than nothing*, or as *arise from the subtraction of a greater quantity from a lesser*, which is an operation evidently impossible to be performed : and, as to the *negative roots* of an equation, they are in truth the real and positive roots of another equation consisting of the same terms as the first equation, but with different signs + and — prefixed to some of them ; so that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, and suppose the proposed, or first, equation to have not only its own proper roots (which they call its *affirmative*, or *positive*, roots,) but to have likewise the roots of a different equation, which they call its *negative* roots. Thus, for example, they would say, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the positive, or affirmative, root, + 16, and the negative root, − 20. But this latter number, 20, is, in truth, the root of a different equation, to wit, of the equation $xx - 4x = 320$. So that this kind of absurd and fantastick language only tends to the confounding together the two different equations $xx + 4x = 320$, and $xx - 4x = 320$, and considering them as if they were one and the same equation. Now this perplexing language is unfortunately used by Mr. *Raphson* in this valuable Treatise, and tends to throw an air of mystery and obscurity upon some of the Problems solved in it, from which they would otherwise have been intirely free. As a proof of the truth of this observation, I shall here insert one of the said Problems, the solution of which is by this means rendered so obscure, that I had a good deal of trouble to find out the meaning of it ; though, if this language had been avoided, and the proper and natural language, belonging to the conditions of the Problem, had been used in its stead, there could not have been the least difficulty in understanding it. This Problem is the 24th, in page 32 of the 2d edition of the book, and is, *verbatim et litteratim*, as follows.

PROBLEMA

PROBLEMA XXIV.

Æquationum Quintæ Poteſtatis Adfectarum Solutio.

Proponatur $-aaaaa + 7aaaa - 20aaa + 155aa = 10{,}000.$

Hoc eſt, $-aaaaa + baaaa - caaa + daa = f.$

$$\text{Theor. } x = \frac{f + ggggg + cggg - bgggg - dgg}{4bggg + 2dg - 5gggg - 3cgg}$$

Sit $g = -5$

$f + ggggg + cggg - bgggg - dgg = -3875$

$4bggg + 2dg - 5gggg - 3cgg = -9675)\ (-3875{,}0\ (+,4 = x$

$-5,$

$+,4$

$g = -4{,}6$

$f + ggggg + cggg - bgggg - dgg = -420{,}36896$

$4bggg + 2dg - 5gggg - 3cgg = -7659{,}736)\ -420{,}36896\ (+,055 = x$

$-4{,}6$

$+,055$

$g = -4{,}545$

$f + ggggg + cggg - bgggg - dgg = -5{,}9603594654 65625$

$4bggg + 2dg - 5gggg - 3cgg = -7410{,}748)\ -5{,}9603594\ (+,00080428 = x$

$-4{,}545$

$+,000{,}804{,}28$

$a = -4{,}544{,}195{,}72$

To this ſolution I have, in my copy of Mr. Raphſon's Tract, ſubjoined the following Note.

Numerus 4.544,195,72 eſt radix æquationis $a^5 + 7a^4 + 20a^3 + 155a^2 = 10{,}000$; quod hic obſcurè innuitur ſub ſpecie radicis negativæ æquationis $-a^5 + 7a^4 - 20a^3 + 155a^2 = 10{,}000$. Omnes ferè difficultates quibus permulti cultioris ingenii viri ab Algebrâ diſcendâ et excolendâ deterrentur, ex hiſce radicibus negativis et aliis quantitatibus negativis,

gativis, seu (ut hodierni Algebræ scriptores absurdè loquuntur,) nihilo minoribus, ortum habent.

In this Problem the letter *a* is used for the unknown quantity, or root of the equation, which is usually denoted by the letter *x*; and the letter *g* is used for the first near value of the root of the equation, which in the two foregoing Tracts has been denoted by the letter *a*; and the letter *x* is used for the difference between *g*, the first near value of the root of the equation, and *a*, its true value, which difference has been denoted in the two foregoing Tracts by the letter *z*. So that, if we express the enunciation of the foregoing Problem in the notation that has been used in the two foregoing Tracts, it will be as follows.

Proponatur $-xxxxx + 7xxxx - 20xxx + 155xx = 10{,}000$,
Sive $-x^5 + 7x^4 - 20x^3 + 155xx = 10{,}000$,

Hoc est, $-xxxxx + bxxxx - cxxx + dxx = f$,
Sive $-x^5 + bx^4 - cx^3 + dx^2 = f$.

$$\text{Theor. } z = \frac{f + aaaaa + caaa - baaaa - daa}{4baaa + 2da - 5aaaa - 3caa},$$

$$\text{or } z = \frac{f + a^5 + ca^3 - ba^4 - da^2}{4ba^3 + 2da - 5a^4 - 3ca^2}.$$

Art. 4. Here, then, the equation proposed by Mr. *Raphson* to be resolved, is said to be $-x^5 + 7x^4 - 20x^3 + 155xx = 10{,}000$, or $155xx - 20x^3 + 7x^4 - x^5 = 10{,}000$. But this is not the equation he resolves; and, indeed, it is not a possible equation, because the greatest possible magnitude of the compound quantity $155xx - 20x^3 + 7x^4 - x^5$ is that which it has when the infinitely small increment of the binomial quantity $20x^3 + x^5$ becomes equal to the contemporary increment of the binomial quantity $155xx + 7x^4$, that is, (if we put $\dot{x}$, or x with a point placed over it, for the infinitely small increment of x,) when $20 \times 3x^2\dot{x} + 5x^4\dot{x}$ becomes equal to $155 \times 2x\dot{x} + 7 \times 4x^3\dot{x}$, or when $60x^2 + 5x^4$ is $= 310x + 28x^3$, or when $60x + 5x^3$ is $= 310 + 28xx$, or when $5x^3 - 28xx + 60x$ is $= 310$, or or when $x^3 - \frac{28xx}{5} + 12x$ is $= 62$, or when $x^3 - 5.6xx + 12x$ is $= 62$; and that is when x is nearly $= 5.5$; at which time the compound quantity $155xx - 20x^3 + 7x^4 - x^5$ will be nearly equal to 2733, as will appear by ſubſtituting 5.5 inſtead of x in the terms of the ſaid quantity $155xx - 20x^3 + 7x^4 - x^5$: and this quantity 2733 (which is the greateſt poſſible magnitude of the compound quantity $155xx - 20x^3 + 7x^4 - x^5$,) is very much leſs than 10,000, or the abſolute term of the equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, and conſequently the ſaid equation is impoſſible. But Mr. *Raphſon*, though he ſets down this equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, as the equation that is to be reſolved, yet really means to reſolve a quite different equation, to wit, the equation that reſults from ſuppoſing x to be a negative quantity, or from ſubſtituting the powers of $-x$, to wit, $+xx$, $-x^3$, $+x^4$, and $-x^5$, in the terms of the ſaid equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$, inſtead of the like powers of $+x$, to wit, $+xx$, $+x^3$, $+x^4$, and $+x^5$; by which ſubſtitution the ſaid equation will be converted into the equation $155 \times +xx - 20 \times -x^3 + 7 \times +x^4 - 1 \times -x^5 = 10,000$, or $155xx + 20x^3 + 7x^4 + x^5 = 10,000$, which is evidently a poſſible equation, and of which the root is 4.544, 195,72, or the ſame number which he obtains by his ſolution of the Problem, and which, with the ſign $-$ prefixed to it, he calls the negative root of the propoſed equation $155xx - 20x^3 + 7x^4 - x^5 = 10,000$. Now all this perplexity would have been avoided, if Mr. *Raphſon* had propoſed at firſt to find the root, or, in the language of modern writers of Algebra, the *affirmative*, or *poſitive*, root, of the equation $155xx + 20x^3 + 7x^4 + x^5 = 10,000$, or $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, which equation is evidently poſſible, and can have only one root. And then all the ſteps of his ſolution would have been clear and eaſy, as will appear by reſolving this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$ according to the principles of his method; which may be done in the manner following.

The

The Resolution of the Affected Equation $x^5 + 7x^4 + 20x^3 + 155x = 10{,}000$, *by Mr.* Raphson's *Method of Approximation.*

Art. 5. In considering this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, it is, in the 1st place, easy to see that x must be greater than 1. For, if we suppose x to be $= 1$, we shall have $xx = 1$, and $x^3 = 1$, and $x^4 = 1$, and $x^5 = 1$; and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ will be $= 1 + 7 + 20 + 155 = 183$; which is very much less than the absolute term 10,000. Therefore 1 must be much less than x.

In the second place, if we suppose x to be $= 10$, we shall have $xx = 100$, and $x^3 = 1000$, and $x^4 = 10{,}000$, and $x^5 = 100{,}000$; so that x^4 alone will be equal to the absolute term 10,000, and consequently $x^5 + 7x^4 + 20x^3 + 155x$ must be very much greater than the said absolute term; and consequently 10 must be much greater than x.

Thirdly, since x is less than 10 and greater than 1, let us suppose it to be equal to 5. Then we shall have $xx = 25$, and $x^3 = 125$, and $x^4 = 625$, and $x^5 = 3125$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 3125 + 7 \times 625 + 20 \times 125 + 155 \times 25 = 3125 + 4375 + 2500 + 3875) = 13{,}875$; which is greater than the absolute term 10,000. Therefore 5 is greater than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

We will therefore, in the 4th place, suppose x to be $= 4$. And then we shall have $xx = 16$, and $x^3 = 64$, and $x^4 = 256$, and $x^5 = 1024$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 1024 + 7 \times 256 + 20 \times 64 + 155 \times 16 = 1024 + 1792 + 1280 + 2480) = 6576$; which is less than the absolute term 10,000. Therefore 4 is less than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

It

It appears therefore that the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, is greater than 4, but lefs than 5. And either of thefe values might very well ferve for a firft near value of the faid root, or for the bafis of a further approximation to it. Mr. Raphfon makes choice of 5, which is greater than the truth.

Art. 6. Let us then fuppofe a, or the firft near value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, to be $= 5$; and let z be the difference by which it exceeds the true value of x. Then will x be $= a - z$, and confequently xx will be $(= \overline{a-z}|^2) = aa - 2az + \&c$, and x^3 will be $(= \overline{a-z}|^3) = a^3 - 3a^2z + \&c$, and x^4 will be $(= \overline{a-z}|^4) = a^4 - 4a^3z + \&c$, and x^5 will be $(= \overline{a-z}|^5) = a^5 - 5a^4z + \&c$. Therefore $x^5 + 7x^4 + 20x^3$

$$+ 155xx \text{ will be } = \left\{\begin{array}{rrl} & & \overline{a^5 - 5a^4z + \&c,} \\ + & 7 \times & \overline{a^4 - 4a^3z + \&c,} \\ + & 20 \times & \overline{a^3 - 3a^2z + \&c,} \\ + & 155 \times & \overline{aa - 2az + \&c,} \end{array}\right\}$$

$$= \left\{\begin{array}{rrcrl} & a^5 & - & 5a^4z & + \&c, \\ + & 7a^4 & - & 28a^3z & + \&c, \\ + & 20a^3 & - & 60a^2z & + \&c, \\ + & 155aa & - & 310az & + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10,000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z - 28a^3z - 60a^2z - 310az + \&c$, will alfo be $= 10,000$, and confequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both fides,) we fhall have $a^5 + 7a^4 + 20a^3 + 155aa = 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$, or (becaufe a is $= 5$, and confequently $a^5 + 7a^4 + 20a^3 + 155aa$ is $= 13,875$, as has been fhewn in art. 5,) we fhall have $13,875 = 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$, and confequently (fubtracting 10,000 from both fides,) $3875 = 5a^4z + 28a^3z + 60a^2z + 310az = z \times \overline{5a^4 + 28a^3 + 60a^2 + 310a}$.

 Therefore

Therefore z will be $= \frac{3875}{5a^4 + 28a^3 + 60a^2 + 310a}$ $(= \frac{3875}{5 \times \overline{5}|^4 + 28 \times \overline{5}|^3 + 60 \times \overline{5}|^2 + 310 \times 5} = \frac{3875}{5 \times 625 + 28 \times 125 + 60 \times 25 + 310 \times 5} = \frac{3875}{3125 + 3500 + 1500 + 1550} = \frac{3875}{9675}) = 0.4$. Therefore $a - z$, or x, will be $(= a - 0.4 = 5.0 - 0.4) = 4.6$; and 4.6 will be a second near value of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

We must next try whether this second near value of x is greater or less than its true value; and for this purpose we must substitute it, instead of x, in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$.

Now, if we suppose x to be $= 4.6$, we shall have xx $(= \overline{4.6}|^2) = 21.16$, and x^3 $(= \overline{4.6}|^3) = 97.336$, and x^4 $(= \overline{4.6}|^4) = 447.7456$, and x^5 $(= \overline{4.6}|^5) = 2059.62976$, and $155xx$ $(= 155 \times 21.16) = 3279.80$, and $20x^3$ $(= 20 \times 97.336) = 1946.720$, and $7x^4$ $(= 7 \times 447.7456) = 3134.2192$, and consequently $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 2059.62976 + 3134.2192 + 1946.720 + 3279.80) = 10,420.36896$; which is greater than 10,000, or the absolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Therefore 4.6 will be greater than the true value of x in that equation.

Art. 7. To find a third near value of the root of this equation, let a be supposed to be $= 4.6$, and z be the difference by which a, or 4.6, exceeds the true value of the said root.

Then we shall have, as before, $x = a - z$, and consequently xx $(= \overline{a - z}|^2) = aa - 2az + \&c$, and x^3 $(= \overline{a - z}|^3) = a^3 - 3a^2z + \&c$, and x^4 $(= \overline{a - z}|^4) = a^4 - 4a^3z + \&c$, and x^5 $(= \overline{a - z}|^5) = a^5 - 5a^4z + \&c$,

and

and $7x^4$ ($= 7 \times \overline{a^4 - 4a^3z + \&c,}$) $= 7a^4 - 28a^3z +$ &c, and $20x^3$ ($= 20 \times \overline{a^3 - 3a^2z + \&c,}$) $= 20a^3 - 60a^2z$ + &c, and $155xx$ ($= 155 \times \overline{aa - 2az + \&c,}$) $= 155aa - 310az$ + &c, and $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{\begin{array}{rcrl} & a^5 - & 5a^4z & + \&c, \\ + & 7a^4 - & 28a^3z & + \&c, \\ + & 20a^3 - & 60a^2z & + \&c, \\ + & 155aa - & 310az & + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000$.

Therefore $\left\{\begin{array}{rcrl} & a^5 - & 5a^4z & + \&c, \\ + & 7a^4 - & 28a^3z & + \&c, \\ + & 20a^3 - & 60a^2z & + \&c, \\ + & 155aa - & 310az & + \&c, \end{array}\right\}$ will likewife be $= 10{,}000$, and confequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both fides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been fhewn in the laft article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $\overline{4.6}|^5 + 7 \times \overline{4.6}|^4 + 20 \times \overline{4.6}|^3 + 155 \times \overline{4.6}|^2$, is $= 10{,}420.36896$.

Therefore $10{,}420.36896$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and confequently (fubtracting $10{,}000$ from both fides of the equation,) 420.36896 will be $= 5a^4z + 28a^3z + 60a^2z + 310az$ ($= 5 \times \overline{4.6}|^4 \times z + 28 \times \overline{4.6}|^3 \times z + 60 \times \overline{4.6}|^2 \times z + 310 \times 4.6 \times z = 5 \times 447.7456 \times z + 28 \times 97.336 \times z + 60 \times 21.16 \times z + 310 \times 4.6 \times z = 2238.7280 \times z + 2725.408 \times z + 1269.60 \times z + 1426.0 \times z$) $= 7659.7360 \times z$, and confequently z will be ($= \frac{420.36896}{7659.7360}$) $= 0.0548$, or nearly 0.055. Therefore x, or $a - z$, or $4.6 - z$, will be nearly ($= 4.6 - 0.055$,) $= 4.545$; and confequently this number 4.545 will be a third near value of the root of the propofed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Q. E. I.

 Now

Now let this number 4.545 be ſubſtituted inſtead of x in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$, in order to diſcover whether the reſult will be greater, or leſs, than 10,000, or the abſolute term of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Now, if x be ſuppoſed to be $= 4.545$, we ſhall have xx $(= \overline{4.545}|^2) = 20.657{,}025$, and x^3 $(= \overline{4.545}|^3) = 93.886{,}178{,}625$, and x^4 $(= \overline{4.545}|^4) = 426.712{,}681{,}850{,}625$, and x^5 $(= \overline{4.545}|^5) = 1939.409{,}139{,}011{,}090{,}625$, and conſequently $7x^4$ $(= 7 \times 426.712{,}681{,}850{,}625) = 2986.988{,}772{,}954{,}375$, and $20x^3$ $(= 20 \times 93.886{,}178{,}625) = 1877.723{,}572{,}500$, and $155xx$ $(= 155 \times 20.657{,}025) = 3201.838{,}875$, and $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 1939.409{,}139{,}011{,}090{,}625 + 2986.988{,}772{,}954{,}375 + 1877.723{,}572{,}500 + 3201.838{,}875) = 10{,}005.960{,}359{,}465{,}465{,}625$; which is greater than 10,000, or the abſolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Therefore 4.545 will be greater than the true value of x in that equation.

Art. 8. To find a fourth near value of the root of this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, let a be ſuppoſed to be $= 4.545$, and z be ſuppoſed to be the difference by which a, or 4.545, exceeds the true value of the ſaid root.

Then we ſhall, as before, have $x = a - z$, and conſequently xx $(= \overline{a - z}|^2) = aa - 2az + \&c$, and x^3 $(= \overline{a - z}|^3) = a^3 - 3a^2z + \&c$, and x^4 $(= \overline{a - z}|^4) = a^4 - 4a^3z + \&c$, and x^5 $(= \overline{a - z}|^5) = a^5 - 5a^4z + \&c$, and $7x^4$ $(= 7 \times \overline{a^4 - 4a^3z + \&c,}) = 7a^4 - 28a^3z + \&c$, and $20x^3$ $(= 20 \times \overline{a^3 - 3a^2z + \&c,}) = 20a^3 - 60a^2z + \&c$, and $155xx$ $(= 155 \times \overline{aa - 2az + \&c,}) = 155aa - 310az + \&c$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{\begin{array}{rcccc} & a^5 & - & 5a^4z & + \&c, \\ + & 7a^4 & - & 28a^3z & + \&c, \\ + & 20a^3 & - & 60a^2z & + \&c, \\ + & 155a^2 & - & 310az & + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10,000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z + \&c, - 28a^3z + \&c, - 60a^2z + \&c, - 310az + \&c$, will likewise be $= 10,000$, and consequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both sides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been shewn in the last article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $\overline{4.545}|^5 + 7 \times \overline{4.545}|^4 + 20 \times \overline{4.545}|^3 + 155 \times \overline{4.545}|^2$, is $= 10,005.960,359,465,465,625$.

Therefore $10,005.960,359,465,465,625$ will be $= 10,000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and consequently (subtracting 10,000 from both sides,) $5.960,359,465,465,625$ will be $= 5a^4z + 28a^3z + 60a^2z + 310az$ ($= 5 \times \overline{4.545}|^4 \times z + 28 \times \overline{4.545}|^3 \times z + 60 \times \overline{4.545}|^2 \times z + 310 \times 4.545 \times z = 5 \times 426.712,681,850,625 \times z + 28 \times 93.886,178,625 \times z + 60 \times 20.657,025 \times z + 310 \times 4.545 \times z = 2133.563,409,253,125 \times z + 2628.813,001,500 \times z + 1239.421,500 \times z + 1408.950 \times z$) $= 7410.747,910,753,125 \times z$. Therefore z will be ($= \frac{5.960,359,465,465,625}{7410.747,910,753,125}$) $= 0.000,804,28$, and x, or $a - z$, or $4.545 - z$, will be ($= 4.545,000,00 - 0.000,804,28$) $= 4.544,195,72$. Therefore $4.544,195,72$ will be a fourth near value of the root of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Q. E. I.

This number 4.544,195,72, agrees with the number found by Mr. Raphson, in all its figures.

Art. 9. The foregoing resolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, has been performed at great

great length, in order to set forth, in as clear a manner as possible, the several reasonings upon which the arithmetical operations used in it are grounded, as well as the said operations themselves. And by so doing the subject is rendered so much easier than in Mr. Raphson's very concise and compressed way of treating it, (in which all the reasonings are dropped, and only the arithmetical operations are exhibited,) that, though the above resolution of the said equation is three, or four, times as long as Mr. Raphson's, yet I am fully persuaded that it may be read and understood in a third, or fourth, part of the time that is necessary to a thorough comprehension of Mr. Raphson's resolution of it; even if he had not puzzled the matter by talking of the negative root of the equation $-x^5 + 7x^4 - 20x^3 + 155xx = 10,000$. But that this may appear the more clearly, I will now repeat the foregoing resolution of this equation in the style and manner of Mr. Raphson, by omitting the several reasonings set forth in the foregoing articles, and making use of a Canon, or Theorem, for the purpose of computing the second, third, and fourth values of z, in the same manner as Mr. Raphson has done.

Art. 10. Since each of the three first successive near values of x, or the root of the proposed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, from which the next near values of it are derived, to wit, the three numbers 5, 4.6, and 4.545, and which are successively denoted by the letter a, is greater than the true value of x in the said equation, or than the root of the said equation, it follows that the second, and third, and fourth near values of x will, each of them, be successively denoted by the residual quantity $a - z$; and consequently, by applying the reasonings used in art. 6, in order to obtain the values of z, and of $a - z$, or x, we shall find that z will be, successively, nearly equal to the value of the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10,000}{5a^4 + 28a^3 + 60a^2 + 310a}$, and, therefore, that $a - z$, or x, will be, successively, nearly equal to the value of the quantity a — the fraction

$a^5 +$

$\frac{a^5 + 7a^4 + 20a^3 + 155a^2 - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$. This, then, is the Theorem, or Canon, by the application of which we are to compute the ſecond, and third, and fourth, near values of $a - z$, or x, after taking 5 for the firſt near value of it, or for the firſt value of a.

Now, if a is $= 5$, we ſhall have $z =$ the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{3875}{9675} = 0.4$. Therefore $a - z$ will be $(= 5 - 0.4) = 4.6$; which will therefore be the ſecond near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Secondly, if a be $= 4.6$, we ſhall have $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{420.36896}{7659.7360} = 0.0548$, or, nearly, 0.055. Therefore $a - z$ will be $(= 4.6 - 0.055) = 4.545$; which will therefore be the third near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Thirdly, if a be $= 4.545$, we ſhall have $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{5.960{,}359{,}465{,}465{,}625}{7410.747{,}910{,}753{,}125} = 0.000{,}804{,}28$. Therefore $a - z$ will be $(= 4.545 - 0.000{,}804{,}28) = 4.544{,}195{,}72$; which will therefore be the fourth near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Q. E. I.

Art. 11. Mr. Raphſon's Canon, or Theorem, for the value of z, is expreſſed more conciſely than the foregoing Theorem, $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$. For he uſes the letters b, c, d, and f, for the co-efficients 7, 20, and 155, of the fourth, third, and ſecond, power of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, and for 10,000, the abſolute term of that equation, reſpectively; which produces the following Canon, or Theorem, for the value of z, to to wit, $z = \frac{a^5 + ba^4 + ca^3 + da^2 - f}{5a^4 + 4ba^3 + 3ca^2 + 2da}$. But it appears to me that, though we may ſeem to gain ſomething in point of brevity by uſing this very general notation, we loſe as much in the article of perſpicuity, which is a matter of much greater importance. However, this latter reſolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, which is expreſſed in Mr. Raphſon's conciſe ſtyle and manner, and the foregoing more explicit reſolution of it in art. 5, 6, 7, and 8, (in which the reaſonings, on which the ſeveral arithmetical operations are grounded, are diſtincty ſet forth and repeated,) are, both of them, the ſame in ſubſtance, and are, as I believe, the very beſt method that can be taken for diſcovering the root of the ſaid equation.

Art. 12. It has been obſerved above in art. 2, that Sir Iſaac Newton's method of reſolving numeral equations by approximation differed but little from Mr. Raphſon's, both methods being founded on the ſame principle of conſidering the new, or transformed, equation, (reſulting from the ſubſtitution of $a + z$, or $a - z$, inſtead of x, in the original equation,) as a mere ſimple equation, or neglecting, or omitting, all the terms of it which involved in them any higher power of z than its ſimple power; which reduces the reſolution of all equations, of whatever orders, to the reſolution of a ſimple equation, or, rather, to the reſolution of ſeveral ſucceſſive ſimple equations, by which we make continual approaches to the true value of the root of the original equation. In this grand principle Sir Iſaac Newton's method and Mr. Raphſon's method perfectly agree; and, in finding the ſecond near value of x, or in making the firſt approximation to the true value of x, after having obtained, by conjecture, or trial, or in ſome other manner, the value of what has been here called a, or a firſt near value of x, or the root ſought, there is not the ſmalleſt difference between them. But in the inveſtigation of the third, and fourth, and other following near values of x, there is a little difference in their manner of proceeding, which the reader may be glad to ſee examined. I ſhall therefore now compare

pare the two methods together, in the caſe of a very eaſy equation, by which Sir Iſaac Newton himſelf has thought proper to illuſtrate his method.

A Compariſon between Sir Iſaac Newton's *and Mr.* Raphſon's *Methods of Reſolving Numeral Equations by Approximation.*

Art. 13. Sir Iſaac Newton's Method of Reſolving Numeral Equations by Approximation, is explained by himſelf in his curious little Tract, intitled, *Analyſis per Æquationes Numero Terminorum Infinitas*, (which was written in the year 1666, and communicated to Dr. Iſaac Barrow, and to Mr. John Collins, and to other learned men of that time, in the year 1669,) by an example; which is as follows.

Art. 14. Let it be required to reſolve the cubick equation $x^3 - 2x = 5$.

Here, in the firſt place, it is eaſy to ſee that x is ſomewhat greater than 2, but much leſs than 3. For, if x is taken equal to 2, we ſhall have $2x = 4$, and $x^3 = 8$, and conſequently $x^3 - 2x\ (= 8 - 4) = 4$; which is leſs than 5, or the true value of $x^3 - 2x$ in the propoſed equation: and, if x is taken equal to 3, we ſhall have $2x = 6$, and $x^3 = 27$, and conſequently $x^3 - 2x\ (= 27 - 6) = 21$; which is very much greater than 5, or the true value of $x^3 - 2x$ in the propoſed equation. Therefore the true value of x in that equation muſt be much leſs than 3, and a little greater than 2. Let it therefore be ſuppoſed to be equal to the quantity $2 + z$, in which z denotes the unknown quantity by which the true value of x exceeds 2. And let $2 + z$ be ſubſtituted, inſtead of x, in the propoſed equation $x^3 - 2x = 5$. This may be done as follows.

Since x is $= 2 + z$, we ſhall have $x^3\ (= \overline{2}|^3 + 3 \times \overline{2}|^2$

$\times z$

$\times z + 3 \times 2 \times zz + z^3 = 8 + 3 \times 4 \times z + 3 \times \overline{2zz} + z^3) = 8 + 12z + 6zz + z^3$, and $2x\ (= 2 \times \overline{2 + z}) = 4 + 2z$, and consequently $x^3 - 2x\ (= 8 + 12z + 6zz + z^3 - 4 - 2z) = 4 + 10z + 6zz + z^3$. But $x^3 - 2x$ is $= 5$. Therefore $4 + 10z + 6zz + z^3$ will also be $=$ 5, and consequently (subtracting 4 from both sides,) $10z + 6zz + z^3$ will be $= 1$; and, (subtracting $6zz + z^3$ from both sides,) $10z$ will be $= 1 - 6zz - z^3$. Therefore z will be $= \frac{1 - 6zz - z^3}{10} = \frac{1}{10} - \frac{6zz - z^3}{10} = 0.1 \frac{-6zz - z^3}{10}$, that is, z is less than $\frac{1}{10}$, or 0.1, by the quantity $\frac{6zz + z^3}{10}$. Therefore x, or $2 + z$, is less than $2 + \frac{1}{10}$, or $2 + 0.1$, or 2.1, by the said quantity $\frac{6zz + z^3}{10}$; which, on account of the smallness of z, (which is less than $\frac{1}{10}$,) will be a very small quantity in comparison of z, or of $\frac{1}{10}$, and, *à fortiori*, in comparison of 2, and consequently may be neglected. And therefore 2.1 will be a second near value of x, or the root of the proposed equation $x^3 - 2x = 5$, that will be a little greater than its true value, but nearer to it than any other number that consists of only two places of figures.

Q. E. I.

This is the first step of Sir Isaac Newton's approximation to the root of the equation $x^3 - 2x = 5$, after the assumption of the number 2, by conjecture and trial, for its first near value. And in this first step of the approximation Sir Isaac Newton's and Mr. Raphson's methods exactly co-incide.

Art. 15. But in the next step of the approximation to the value of x, in the said equation $x^3 - 2x = 5$, the two methods are somewhat different from each other, though the number of new figures of the true value of x, that are exact in the next near values of it resulting from both methods,

thods, is the ſame. The difference between the methods in this ſecond ſtage of the approximation is as follows.

Mr. Raphſon corrects the value of x, or the root of the original equation $x^3 - 2x = 5$, already found, to wit, 2.1, (and which is known to be ſomewhat greater than the truth,) by ſubtracting from it the unknown quantity by which it exceeds x, and which we may call v, and ſubſtituting $2.1 - v$ inſtead of x in the ſaid original equation, $x^3 - 2x = 5$, whereby it is transformed into another cubick equation, in which v will be the only unknown quantity; and then he finds a near value of v by reſolving the ſaid transformed equation as if it were only a ſimple equation, or by neglecting the terms which involve the ſquare and cube of v, on account of their ſmallneſs, juſt as we before neglected the terms $6zz$ and z^3 in the foregoing transformed equation $10z + 6zz + z^3 = 1$ for the ſame reaſon. But Sir Iſaac Newton takes no further notice of the original equation $x^3 - 2x = 5$, till he has compleated the whole proceſs of his approximation; but, inſtead of the ſaid original equation, he conſiders the former transformed equation, $10z + 6zz + z^3 = 1$, which was derived from it, and inveſtigates the value of its root, z, to a greater degree of exactneſs than that to which it was before obtained. And this he does in the manner following.

Since it has been ſeen that z is leſs than 0.1, let the quantity by which 0.1 exceeds it be called v, ſo that z ſhall be $= 0.1 - v$; and let $0.1 - v$ be ſubſtituted, inſtead of z, in the transformed equation $10z + 6zz + z^3 = 1$. This may be done as follows.

Since z is $= 0.1 - v$, we ſhall have

$$zz\ (= \overline{0.1 - v}|^2) = 0.01 - 0.2v + vv,$$

and $z^3\ (= \overline{0.1 - v}|^3 = 0.001 - 3 \times 0.01 \times v + 3 \times 0.1 \times vv - v^3) = 0.001 - 0.03v + 0.3vv - v^3$,

and $10z\ (= 10 \times 0.1 - v) = 1 - 10v$,

and $6zz = \quad 0.06 - 1.2v + 6vv$,

 and

and consequently

$$\left\{\begin{array}{r} 10z \\ +\ 6zz \\ +\ z^3 \end{array}\right\} = \left\{\begin{array}{l} 1.00 - 10v \\ + 0.06 - 1.2v + 6vv \\ + 0.001 - 0.03v + 0.3vv - v^3 \end{array}\right\}$$
$$= 1.061 - 11.23v + 6.3vv - v^3.$$

But $10z + 6zz + z^3$ is $= 1$.

Therefore $1.061 - 11.23v + 6.3vv - v^3$ will likewise be $= 1$. And consequently (adding $11.23v$ to both sides,) we shall have $1.061 + 6.3vv - v^3 = 1 + 11.23v$; and, (subtracting 1 from both sides,) we shall have $0.061 + 6.3vv - v^3 = 11.23v$, and (neglecting $6.3vv$ and v^3 as inconsiderable in comparison of 0.061 and $11.23v$) we shall have $0.061 = 11.23v$, or $11.23v = 0.061$; and consequently (dividing both sides by 11.23,) we shall have v ($= \frac{0.061}{11.23}$) $= 0.0054$. Therefore z, or $0.1 - v$, will be ($= 0.1 - 0.0054$) $= 0.0946$, and consequently x, or $2 + z$, will be ($= 2 + 0.0946$) $= 2.0946$. Q. E. I.

In this manner Sir Isaac Newton finds the root of the proposed equation $x^3 - 2x = 5$ to be equal to 2.0946, which is as near the truth as five figures can express it.

Art. 16. He then carries the investigation one step further, by which he obtains the value of x exact to nine places of figures; and for this purpose he proceeds in the manner following.

The last transformed equation was $11.23v = 0.061 + 6.3vv - v^3$; from which it follows that v is accurately equal to $\frac{0.061}{11.23} + \frac{6.3vv - v^3}{11.23}$, or $0.0054 + \frac{6.3vv - v^3}{11.23}$, which is greater than 0.0054 alone, because $6.3vv$ is greater than v^3. Since, therefore, v is greater than 0.0054, let us suppose it to be $= 0.0054 + w$; and let this binomial quantity be substituted, instead of v, in the last transformed equation $11.23v = 0.061 + 6.3vv - v^3$, or, rather, in the equation

equation $11.23v - 6.3vv + v^3 = 0.061$, consisting of the same terms as the former, but in which the terms involving the unknown quantity v are all brought to the same side of the equation, and ranged according to the powers of v, beginning from its lowest power, or the simple power of v. This may be done in the manner following.

Since v is $= 0.0054 + w$, we shall have

$$vv\ (= \overline{0.0054 + w}|^2 = \overline{0.0054}|^2 + 2 \times 0.0054 \times w + w^2) = 0.000,029,16 + 0.0108 \times w + w^2,$$

and

$$v^3\ (= \overline{0.0054 + w}|^3 = \overline{0.0054}|^3 + 3 \times \overline{0.0054}|^2 \times w + 3 \times 0.0054 \times w^2 + w^3$$
$$= 0.000,000,157,464 + 3 \times 0.000,029,16 \times w + 0.0162 \times w^2 + w^3)$$
$$= 0.000,000,157,464 + 0.000,087,48 \times w + 0.0162 \times w^2 + w^3,$$

and

$$11.23v\ (= 11.23 \times \overline{0.0054 + w}) = 0.060,642 + 11.23 \times w,$$

and

$$6.3vv\ (= 6.3 \times \overline{0.000,029,16 + 0.0108 \times w + w^2}) = 0.000,183,708 + 0.068,04 \times w + 6.3ww;$$

and consequently $11.23v - 6.3vv + v^3$ will be $=$

$$\left\{\begin{array}{l} 0.060,642 + 11.23 \times w \\ - 0.000,183,708 - 0.068,04 \times w - 6.3\,ww \\ + 0.000,000,157,464 + 0.000,087,48 \times w + 0.0162w^2 + w^3) \end{array}\right\}$$

$$= \left\{\begin{array}{l} 0.060,642,157,464 + 11.230,087,48 \times w + 0.0162w^2 + w^3 \\ - 0.000,183,708 \quad - 0.068,04 \times w \quad - 6.3 \times w^2 \end{array}\right\}$$

$$= 0.060,458,449,464 + 11.162,047,48w - 6.2838w^2 + w^3.$$

But $11.23v - 6.3vv + v^3$ is $= 0.061$.

Therefore $0.060,458,449,464 + 11.162,047,48 \times w - 6.2838 \times ww + w^3$ will likewise be $= 0.061$; and consequently (subtracting 0.060,458,449,464 from both sides,) $11.162,047,48 \times w - 6.2838ww + w^3$ will be $(= 0.061,000,000,000 - 0.060,458,449,464) = 0.000,541,550,536$; and (neglecting the terms $6.2838ww$ and w^3, as inconsiderable in comparison of $11.162,047,48 \times w$,) we shall have $11.162,047,48 \times w = 0.000,541,550,536$, and conse-

conſequently w $(= \frac{0.000,541,550,536}{11.162,047,48} = 0.000,048,52.$

Therefore v, or $0.0054 + w$, will be $(= 0.0054 + 0.000,048,52) = 0.005,448,52$, and z, or $0.1 - v$, will be $(= 0.100,000,00 - 0.005,448,58) = 0.094,551,48$, and x, or $2 + z$, will be $(= 2 + 0.094,551,48) = 2.094,551,48$; that is, the root of the propoſed equation $x^3 - 2x = 5$ will be $= 2.094,551,48$. Q. E. I.

This number 2.094,551,48 is exact in all the figures, as will be ſhewn in a ſubſequent article.

Art. 17. Having thus ſet forth Sir Iſaac Newton's method of inveſtigating the root of the propoſed equation $x^3 - 2x = 5$ to nine places of figures, we muſt now perform the ſame thing by Mr. Raphſon's method, in order to make a compariſon between the neceſſary operations of the two methods.

Now Mr. Raphſon's method of approximating further to the root of the equation $x^3 - 2x = 5$, after having found it to be equal to $2 + 0.1 - \frac{6zz - z^3}{10}$, or to be ſomewhat leſs than 2.1, is to put v for the unknown quantity by which it falls ſhort of 2.1, and then to ſubſtitute the reſidual quantity $2.1 - v$ in the terms of the original equation $x^3 - 2x = 5$, whereby the ſaid equation will be transformed into another cubick equation, in which v will be the only unknown quantity: and then he determines the value of v by reſolving the ſaid transformed equation as if it was a mere ſimple equation, or by neglecting the terms in which the ſquare or the cube of v occur. This may be done in the manner following.

Since x is $= 2.1 - v$, we ſhall have xx $(= \overline{2.1 - v}|^2 = \overline{2.1}|^2 - 2 \times 2.1 \times v + \&c) = 4.41 - 4.2v + \&c$, and x^3 $(= \overline{2.1 - v}|^3 = \overline{2.1}|^3 - 3 \times \overline{2.1}|^2 \times v + \&c = 9.261 - 3 \times 4.41 \times v + \&c) = 9.261 - 13.23 \times v + \&c$,

&c, and $2x$ ($= 2 \times \overline{2.1 - v}$) $= 4.2 - 2v$, and conſequently $x^3 - 2x$ ($= 9.261 - 13.23 \times v +$ &c $- 4.2 + 2v$) $= 5.061 - 11.23 \times v$ &c.

But $x^3 - 2x$ is $= 5$.

Therefore $5.061 - 11.23 \times v$ &c, will likewiſe be $= 5$, and conſequently (adding $11.23 \times v$ to both ſides,) we ſhall have $5.061 = 5 + 11.23 \times v$, and (ſubtracting 5 from both ſides,) we ſhall have $11.23 \times v = 0.061$, and conſequently v ($= \frac{0.061}{11.23}$) $= 0.0054$. Therefore x, or $2.1 - v$, will be ($= 2.1 - 0.0054$) $= 2.0946$; or 2.0946 will be a third near value of the root of the propoſed equation $x^3 - 2x = 5$. Q. E. I.

This third near value of x is the very ſame with the third near value of it obtained above, in art. 15, by Sir Iſaac Newton's method.

Art. 18. In this ſtep of the approximation, by which we obtain the number 2.0946 for the third near value of the root of the propoſed equation $x^3 - 2x = 5$, the principal difference between the two methods ſeems to conſiſt in this, to wit, that by Mr. Raphſon's method we are obliged to raiſe the two firſt terms of the powers of the compound quantity $2.1 - v$, and conſequently to raiſe the powers of the number 2.1, which conſiſts of two figures; whereas in Sir Iſaac Newton's method of proceeding, we had occaſion only to raiſe the powers of the compound quantity $0.1 - v$, and conſequently to raiſe the powers of the number 0.1, which conſiſts of only one figure; which is ſomewhat eaſier than to raiſe the powers of 2.1. But both operations are ſo eaſy, that the difference of the labour of performing them is hardly worth conſidering. And, with reſpect to the ſimplicity of conception in the two methods, Mr. Raphſon's method ſeems to be preferable to Sir Iſaac Newton's; becauſe the former always refers to the original equation $x^3 - 2x = 5$, whereas the latter method refers to the preceeding transformed equation $10z + 6zz + z^3 = 1$, which has more

more terms and larger co-efficients than the original equation $x^3 - 2x = 5$.

Art. 19. But in the next step of the approximation by Mr. Raphson's method, we shall find the labour of raising the powers of the value of x already found, to wit, the powers of 2.0946, to be considerably greater than that of raising the powers of the last preceeding supplement of it according to Sir Isaac Newton's method, that supplement being only the decimal fraction 0.0054, in which there are only two significant figures. This will appear by performing this step of the approximation by Mr. Raphson's method; which may be done as follows.

Art. 20. The last near value we found for x, or the root of the equation $x^3 - 2x = 5$, by Mr. Raphson's method, was 2.0946. Now this near value of x is greater than its true value. For, if we suppose x to be $= 2.0946$, we shall have x^3 ($= \overline{2.0946}|^3$) $= 9.189,741,550,536$, and $2x$ ($= 2 \times 2.0946$) $= 4.1892$, and consequently $x^3 - 2x$ ($= 9.189,741,550,536 - 4.1892$) $= 5.000,541,550,536$; which is greater than 5, or the absolute term of the equation $x^3 - 2x = 5$: and consequently 2.0946 must be greater than the true value of the root of the said equation.

We will therefore suppose x to be $= 2.0946 - w$, and substitute this residual quantity instead of x in the terms of the equation $x^3 - 2x = 5$.

Now, since x is $= 2.0946 - w$, we shall have xx ($= \overline{2.0946 - w}|^2 = \overline{2.0946}|^2 - 2 \times 2.0946 \times w + \&c) = 4.387,349,16 - 4.1892 \times w + \&c$, and x^3 ($= \overline{2.0946 - w}|^3 = \overline{2.0946}|^3 - 3 \times \overline{2.0946}|^2 \times w + \&c = 9.189,741,550,536 - 3 \times 4.387,349,16 \times w + \&c) = 9.189,741,550,536 - 13.162,047,48 \times w + \&c$, and $2x$ ($= 2 \times \overline{2.0946 - w}$) $= 4.1892 - 2w$, and consequently $x^3 - 2x =$

$$\left\{ \begin{array}{lcl} 9.189,741,550,536 & - & 13.162,047,48 \times w + \&c \\ -4.189,2 & + & 2.000,000,00 \times w \end{array} \right\}$$

$= 5.000,541,550,536 - 11.162,047.48 \times w + \&c.$

But

But $x^3 - 2x$ is $= 5$.

Therefore $5,000,541,550,536 - 11.162,047,48 \times w +$ &c, will be $= 5$; and consequently (adding $11.162,047,48 \times w$ to both fides,) we fhall have $5.000,541,550,536 = 5 + 11.162,047,48 \times w$, and (fubtracting 5 from both fides,) $0.000,541,550,536 = 11.162,047,48 \times w$, or $11.162,047,48 \times w = 0.000,541,550,536$. Therefore w will be $\left(= \frac{0.000,541,550,536}{11.162,047,48}\right) = 0.000,048,52$; and confequently x, or $2.0946 - w$, will be $(= 2.094,600,00 - 0.000,048,52) = 2.094,551,48$. Therefore 2.094,551,48 will be a fourth near value of x, or the root of the propofed equation $x^3 - 2x = 5$. Q. E. I.

This fourth near value of x is the very fame with the fourth near value of it obtained above, in art. 16, by Sir Ifaac Newton's method.

Art. 21. In this laft ftage of Mr. Raphfon's approximation to the root of the propofed equation $x^3 - 2x = 5$, we have been obliged to raife the powers of the number 2.0946, which confifts of five places of figures; whereas in Sir Ifaac Newton's way of proceeding we only raifed the powers of the decimal fraction 0.0054, which contains only two fignificant figures. But then in that way of proceeding we were obliged to multiply v, or $0.0054 + w$, into 11.23, and vv, or $0.000,029,16 + 0.0108 \times w + w^2$, into 6.3; whereas in Mr. Raphfon's way of proceeding we have only to multiply x, or $2.0946 - w$, into the very fimple co-efficient 2. So that, upon the whole, the difference of the labour of computation in the two methods is not very confiderable, though it is rather lefs in Sir Ifaac Newton's method than in Mr. Raphfon's. But in point of fimplicity of conception Mr. Raphfon's method feems much fuperiour to Sir Ifaac's, becaufe it never lofes fight of the original equation $x^3 - 2x = 5$, which is to be refolved.

And, further, we may obferve, in favour of Mr. Raphfon's method, that it never requires us to raife any more than the two first terms of the binomial and residual quantities $2 + z$, and $2.1 - v$, and $2.0946 - w$, which are substituted instead of x in the original equation $x^3 - 2x = 5$; whereas in Sir Isaac Newton's method it is necessary to raise the other terms of the binomial and residual quantities $2 + z$, and $0.1 - v$, and $0.0054 + w$; which increases the number and intricacy of the operations of the investigation. And therefore, upon the whole, I consider Mr. Raphson's method of approximating to the values of the roots of such equations as preferable to Sir Isaac Newton's.

A Proof of the Exactness of the Number 2.094,551,48, *that has been found by the foregoing Methods of Approximation for the Root of the Equation* $x^3 - 2x = 5$.

Art. 22. It remains that we prove the work to have been rightly performed, or that we shew that the last number 2.094,551,48, obtained by both these methods, is a very near value of the root x of the proposed equation $x^3 - 2x = 5$, and that we determine to how many figures it is exact.

Now the plainest and best method of doing this is to substitute the number 2.094,551,48, instead of x, in the compound quantity $x^3 - 2x$, in order to discover whether the quantity resulting from this substitution will be greater, or less, than 5, or the absolute term of the proposed equation $x^3 - 2x = 5$: and, if it shall appear that the said result is greater than 5, we may conclude that the said number 2.094,551,48 is greater than the true value of x in the said equation; and, if it shall appear that the said result is less than 5, we may conclude that the said number is less than the true value of x. And, when this has been thus disco-

discovered, we must, in the next place, endeavour to determine to how many figures this number 2.094,551,48 coincides with the more accurate value of x : and, for this purpose, we must, if this number be less than x, increase it by the addition of an unit in the last place of figures; and, if it be greater than x, we must diminish it by the same small quantity, and then substitute the new number thereby obtained, to wit, 2.094,551,49, or 2.094,551,47, instead of x, in the compound quantity $x^3 - 2x$. And, if it shall appear that the value of that compound quantity resulting from that substitution is greater, or less, than 5, we may conclude that the number 2.094,551,49, or 2.094,551,47, is accordingly greater, or less, than the true value of x, in the equation $x^3 - 2x = 5$, and consequently that the said true value is of an intermediate magnitude between 2.094,551,49 and 2.094,551,48, or between 2.094,551,48 and 2.094,551,47.

Now, if we take $x = 2.094,551,48$, we shall have

$xx = 4.387,145,902,370,190,4$,

and $x^3 = 9.189,102,942,785,417,810,201,792$,

and $2x = 4.189,102,96$,

and consequently

$x^3 - 2x = 4.999,999,982,785,417,810,201,792$; which number is somewhat less than 5, or the absolute term of the proposed equation $x^3 - 2x = 5$. Therefore 2.094,551,48 must be somewhat less than the true value of x in the said equation.

Secondly, since x is greater than 2.094,551,48, we must now compare it with 2.094,551,49, by substituting that number instead of it in the compound quantity $x^3 - 2x$.

Now, if x is taken $= 2.094,551,49$, or $2.094,551,48 + 0.000,000,01$, we shall have x^3 ($= \overline{2.094,551,48}|^3 + 3 \times \overline{2.094,551,48}|^2 \times 0.000,000,01 + 3 \times 2.094,551,48 \times \overline{0.000,000,01}|^2 + \overline{0.000,000,01}|^3 = 9.189,102,942$, &c, +

 3 ×

$3 \times 4.387,145,902$, &c $\times 0.000,000,01 + 0.000,000,000$, &c $+ 0.000,000,000$, &c $= 9.189,102,942$, &c $+ 13.161,437,706$, &c $\times 0.000,000,01 + 0.000.000,000$, &c $+ 0.000,000,000$, &c $= 9.189,102,942$, &c $+ 0.000,000,131$, &c $+ 0.000,000,000$, &c $+ 0.000,000,000$, &c) $= 9.189,103,073$, &c; and $2x$ ($= 2 \times 2.094,551,49$) $= 4.189,102,98$; and consequently $x^3 - 2x$ ($= 9.189,103,07$, &c $- 4.189,102,98$) $= 5.000,000,09$, &c; which is greater than 5. Therefore 2.094,551,49 must be greater than the true value of x in the equation $x^3 - 2x = 5$.

But it has been shewn that 2.094,551,48 is less than the said true value.

Therefore the true value of x in the equation $x^3 - 2x = 5$, will be of an intermediate magnitude between 2.094,551,48 and 2.094,551,49; and consequently all the figures of the number 2.094,551,48, which we found by the foregoing processes of Sir Isaac Newton's and Mr. Raphson's methods of approximation for a fourth near value of the root of the equation $x^3 - 2x = 5$, are exact. Q. E. D.

Of the Difficulty of finding a, *or the First near Value of the Root of an Affected Equation, in certain Cases.*

Art. 23. There is another difficulty that occurs sometimes in resolving high equations by approximation, whether by Sir Isaac Newton's method or by Mr. Raphson's; which indeed are substantially the same. The difficulty I mean, is that of finding the first near value of the root sought (which we have called a in this discourse,) to one, or two places of figures, in order to make it the basis of a further approximation to the true value of the root by either of these methods of approximation. Now, when the equation is known to have but one root, that is, but one real and affirmative root, (for

(for all other roots are not worth confidering,) this difficulty will not be great; becaufe it will always be eafy to find a tolerably near value of the root by conjectures and trials, and particularly by fuppofing x, or the root of the propofed equation, firft, to be equal to 1, and 2dly, to be $= 10$, and 3dly, to be equal to fome fhort intermediate number confifting of only one figure, or, if the root appears to be greater than 10, by fuppofing it to be equal to 100, or 1000, and afterwards fuppofing it to be equal to fome fhort intermediate number confifting of two figures; as was done above in art. 5, in finding the firft near value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. But, when the equation confifts of terms connected together partly by the fign $+$, and partly by the fign $-$, and confequently it may, for aught we know to the contrary, have two, or three, or four, or more real and affirmative roots, which may be of very different magnitudes, the aforefaid method of conjectures and trials (though by no means ufelefs,) is lefs expeditious and fatisfactory in affifting us to find the firft near value of one of the roots than in the former cafe; and we are often puzzled to know which of the roots it would be moft expedient to begin to inveftigate. Now, in moft of thefe cafes, I believe, it will be advifeable to begin by inveftigating the leaft root, and for that purpofe to expunge from the equation all the terms that have the fign $-$ prefixed to them, and to find, to about two places, or, at moft, to three places, of figures, the root of the remaining equation. For this root will always be lefs than the leaft root of the original equation, if it really has (as it appears to have,) more than one real and affirmative root; or it will be lefs than the only root of the original equation, if (notwithftanding the appearances to the contrary,) it really has but one root. When the root of this fecond, or curtailed, equation, has been difcovered, it may be called a, and made the ground-work of an approximation to the leaft root of the original equation, and the binomial quantity $a + z$ may be fubftituted in the original equation inftead of x, and the transformed equation thence arifing may be refolved as if it was a mere fimple equation, agreeably to Mr. Raph-

fon's

ſon's method of approximation; and the value of z thereby obtained, being added to a, will give us a known value of $a + z$, or a ſecond near value of the leaſt, or the only, root of the propoſed equation: after which we may proceed to find the ſaid leaſt, or only, root of the propoſed equation by a further proſecution of Mr. Raphſon's method of approximation above-deſcribed. This method of finding a firſt near value, a, of the leaſt root of a propoſed equation that ſeems to have more than one real and affirmative root, is explained more at length in the third volume of the Collection of Mathematical Tracts, called *Scriptores Logarithmici*, in my Diſcourſe on the Reverſion of Infinite Serieſes publiſhed in that Volume; to which I refer the reader. See the ſaid 3d Volume, pages 724, 725, 726, 727, &c, - - - to page 761. And, with this improvement of it in the caſe of equations that have, or ſeem to have, more than one real and poſitive root, I believe it may ſafely be affirmed that Mr. Raphſon's Method of Reſolving Affected Equations is the beſt General Method of effecting that purpoſe in all equations above quadraticks that has hitherto been diſcovered.

End of the Obſervations on Mr. Raphſon's *Method of Reſolving Affected Equations by Approximation.*

A

T A B L E

OF THE

SQUARE AND CUBE ROOTS OF THE NATURAL NUMBERS 1, 2, 3, 4, 5, &c, to 180;

Being Table XIX. *of Mr.* James Dodſon's *valuable Tables of Computation, intitled* The Calculator, *that were publiſhed in the Year* 1747.

No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.
1	1.000,000	1.000,000	31	5.567,764	3.141,381	61	7.810,250	3.936,497
2	1.414,214	1.259,921	32	5.656,854	3.174,802	62	7.874,008	3.957,892
3	1.732,051	1.442,250	33	5.744,563	3.207,534	63	7.937,254	3.979,057
4	2.000,000	1.587,401	34	5.830,952	3.239,612	64	8.000,000	4.000,000
5	2.236,068	1.709,976	35	5.916,080	3.271,066	65	8.062,258	4.020,726
6	2.449,490	1.817,121	36	6.000,000	3.301,927	66	8.124,038	4.041,240
7	2.645,751	1.912,933	37	6.082,763	3.332,222	67	8.185,353	4.061,548
8	2.828,427	2.000,000	38	6.164,414	3.361,975	68	8.246,211	4.081,656
9	3.000,000	2.080,084	39	6.244,998	3.391,211	69	8.306,624	4.101,566
10	3.162,278	2.154,435	40	6.324,555	3.419,952	70	8.366,600	4.121,285
11	3.316,625	2.223,980	41	6.403,124	3.448,217	71	8.426,150	4.140,818
12	3.464,102	2.289,428	42	6.480,741	3.476,027	72	8.485,281	4.160,168
13	3.605,551	2.351,335	43	6.557,439	3.503,398	73	8.544,004	4.179,339
14	3.741,657	2.410,142	44	6.633,250	3.530,348	74	8.602,325	4.198,336
15	3.872,983	2.466,212	45	6.708,204	3.556,893	75	8.660,254	4.217,163
16	4.000,000	2.519,842	46	6.782,330	3.583,048	76	8.717,798	4.235,824
17	4.123,106	2.571,282	47	6.855,655	3.608,826	77	8.774,964	4.254,321
18	4.242,641	2.620,741	48	6.928,203	3.634,241	78	8.831,761	4.272,659
19	4.358,899	2.668,402	49	7.000,000	3.659,306	79	8.888,194	4.290,841
20	4.472,136	2.714,418	50	7.071,068	3.684,031	80	8.944,272	4.308,870
21	4.582,576	2.758,923	51	7.141,428	3.708,430	81	9.000,000	4.326,749
22	4.690,416	2.802,039	52	7.211,103	3.732,511	82	9.055,385	4.344,481
23	4.795,832	2.843,867	53	7.280,110	3.756,286	83	9.110,434	4.362,071
24	4.898,979	2.884,499	54	7.348,469	3.779,763	84	9.165,151	4.379,519
25	5.000,000	2.924,018	55	7.416,198	3.802,953	85	9.219,544	4.396,830
26	5.099,020	2.962,496	56	7.483,315	3.825,862	86	9.273,618	4.414,005
27	5.196,152	3.000,000	57	7.549,834	3.848,501	87	9.327,379	4.431,047
28	5.291,503	3.036,589	58	7.615,773	3.870,877	88	9.380,832	4.447,960
29	5.385,165	3.072,317	59	7.681,146	3.892,996	89	9.433,981	4.464,745
30	5.477,226	3.107,232	60	7.745,967	3.914,867	90	9.486,833	4.481,405
No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.

No	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.
91	9.539,392	4.497,942	121	11.000,00	4.946,088	151	12.288,21	5.325,074
92	9.591,663	4.514,357	122	11.045,36	4.959,675	152	12.328,83	5.336,803
93	9.643,651	4.530,655	123	11.090,54	4.973,190	153	12.369,32	5.348,481
94	9.695,360	4.546,836	124	11.135,53	4.986,631	154	12.409,67	5.360,108
95	9.746,794	4.562,903	125	11.180,34	5.000,000	155	12.449,90	5.371,685
96	9.797,959	4.578,857	126	11.224,97	5.013,298	156	12.490,00	5.383,213
97	9.848,858	4.594,701	127	11.269,43	5.026,526	157	12.529,96	5.394,690
98	9.899,495	4.610,436	128	11.313,71	5.039,684	158	12.569,81	5.406,120
99	9.949,874	4.626,065	129	11.357,82	5.052,774	159	12.609,52	5.417,501
100	10.000,00	4.641,589	130	11.401,75	5.065,797	160	12.649,11	5.428,835
101	10.049,88	4.657,010	131	11.445,52	5.078,753	161	12.688,58	5.440,122
102	10.099,50	4.672,330	132	11.489,12	5.091,643	162	12.727,92	5.451,362
103	10.148,89	4.687,548	133	11.532,56	5.104,469	163	12.767,15	5.462,556
104	10.198,04	4.702,669	134	11.575,84	5.117,230	164	12.806,25	5.473,703
105	10.246,95	4.717,694	135	11.618,95	5.129,928	165	12.845,23	5.484,806
106	10.295,63	4.732,624	136	11.661,90	5.142,563	166	12.884,10	5.495,865
107	10.344,08	4.747,459	137	11.704,70	5.155,137	167	12.922,85	5.506,879
108	10.392,30	4.762,203	138	11.747,34	5.167,649	168	12.961,48	5.517,848
109	10.440,31	4.776,856	139	11.789,83	5.180,101	169	13.000,00	5.528,775
110	10.488,09	4.791,420	140	11.832,16	5.192,494	170	13.038,40	5.539,658
111	10.535,65	4.805,896	141	11.874,34	5.204,828	171	13.076,70	5.550,499
112	10.583,01	4.820,284	142	11.916,38	5.217,103	172	13.114,88	5.561,298
113	10.630,15	4.834,588	143	11.958,26	5.229,321	173	13.152,95	5.572,054
114	10.677,08	4.848,808	144	12.000,00	5.241,482	174	13.190,91	5.582,770
115	10.723,81	4.862,944	145	12.041,59	5.253,588	175	13.228,76	5.593,445
116	10.770,33	4.876,999	146	12.083,06	5.265,637	176	13.266,50	5.604,079
117	10.816,65	4.890,973	147	12.124,36	5.277,632	177	13.304,13	5.614,673
118	10.862,78	4.904,868	148	12,165,53	5.289,572	178	13.341,66	5.625,226
119	10.908,71	4.918,685	149	12.206,56	5.301,459	179	13.379,09	5.635,741
120	10.954,45	4.932,424	150	12.247,45	5.313,293	180	13.416,41	5.646,216
No	Sq. Root	Cube Rt.	No.	Sq. Root.	Cube Rt.	No.	Sq. Root.	Cube Rt.

A

TABLE

OF THE

SQUARE-ROOTS AND RECIPROCALS OF ALL NUMBERS,

From 1 to 1000.

Computed by Dr. CHARLES HUTTON, *Professor of Mathematicks at the Royal Military Academy at Woolwich in Kent.*

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
1	1.0	1.000,000,000,0	51	0.019,607,8	7.141,428,428,5
2	0.5	1.414,213,562,4	52	0.019,230,8	7.211,102,550,9
3	0.333,333,3	1.733,050,807,6	53	0.018,867,9	7.280,109,889,3
4	0.25	2.000,000,000,0	54	0.018,518,5	7.348,469,228,3
5	0.2	2.236,067,977,5	55	0.018,181,8	7.416,198,487,1
6	0.166,666,6	2.449,489,742,8	56	0.017,857,1	7.483,314,773,5
7	0.142,857,1	2.645,751,311,1	57	0.017,543,9	7.549,834,435,3
8	0.125	2.828,427,124,7	58	0.017,241,4	7.615,773,105,9
9	0.111,111,1	3.000,000,000,0	59	0.016,949,0	7.681,145,747,9
10	0.1	3.162,277,660,2	60	0.016,666,6	7.745,966,692,4
11	0.090,909,0	3.316,624,790,4	61	0.016,393,4	7.810,249,675,9
12	0.083,333,3	3.464,101,615,1	62	0.016,129,0	7.874,007,874,0
13	0.076,923,0	3.605,551,275,5	63	0.015,873,0	7.937,253,933,2
14	0.071,428,5	3.741,657,386,8	64	0.015,625,	8.000,000,000,0
15	0.066,666,6	3.872,983,346,2	65	0.015,384,6	8.062,257,748,3
16	0.062,5	4.000,000,000,0	66	0.015,151,5	8.124,038,404,6
17	0.058,823,5	4.123,105,625,6	67	0.014,925,4	8.185,352,771,9
18	0.055,555,5	4.242,640,687,1	68	0.014,705,9	8.246,211,251,2
19	0.052,631,6	4.358,898,943,5	69	0.014,492,8	8.306,623,862,9
20	0.05	4.472,135,955,0	70	0.014,285,7	8.366,600,265,3
21	0.047,619,0	4.582,575,695,0	71	0.014,084,5	8.426,149,773,2
22	0.045,454,5	4.690,415,759,8	72	0.013,888,8	8.485,281,374,2
23	0.043,478,3	4.795,831,523,3	73	0.013,698,6	8.544,003,745,3
24	0.041,666,6	4.898,979,485,6	74	0.013,513,5	8.602,325,267,0
25	0.04	5.000,000,000,0	75	0.013,333,3	8.660,254,037,8
26	0.038,461,5	5.099,019,513,6	76	0.013,157,9	8.717,797,887,1
27	0.037,037,0	5.196,152,422,7	77	0.012,987,0	8.774,964,387,4
28	0.035,714,3	5.291,502,622,1	78	0.012,820,5	8.831,760,866,3
29	0.034,482,8	5.385,164,807,1	79	0.012,658,2	8.888,194,417,3
30	0.033,333,3	5.477,225,575,1	80	0.012,5	8.944,271,910,0
31	0.032,258,1	5.567,764,362,8	81	0.012,345,7	9.000,000,000,0
32	0.031,25	5.656,854,249,5	82	0.012,195,0	9.055,385,138,1
33	0.030,303,0	5.744,562,646,5	83	0.012,048,2	9.110,433,579,1
34	0.029,411,8	5.830,951,894,8	84	0.011,904,8	9.165,251,389,9
35	0.028,571,4	5.916,079,783,1	85	0.011,764,7	9.219,544,457,3
36	0.027,777,7	6.000,000,000,0	86	0.011,627,9	9.273,618,495,5
37	0.027,027,0	6.082,762,530,3	87	0.011,494,3	9.327,379,053,1
38	0.026,315,8	6.164,414,003,0	88	0.011,363,6	9.380,831,519,6
39	0.025,641,0	6.244,997,998,4	89	0.012,236,0	9.433,981,132,1
40	0.025,	6.324,555,320,3	90	0.011,111,1	9.486,832,980,5
41	0.024,390,2	6.403,124,237,4	91	0.010,989,0	9.539,392,014,2
42	0.023,809,5	6.480,740,698,4	92	0.010,869,6	9.591,663,046,6
43	0.023,255,8	6.557,438,524,3	93	0.010,752,7	9.643,650,761,0
44	0.022,727,2	6.633,249,580,7	94	0.010,638,3	9.695,359,714,8
45	0.022,222,2	6.708,203,932,5	95	0.010,526,3	9.746,794,344,8
46	0.021,739,1	6.782,329,983,1	96	0.010,416,6	9.797,958,971,1
47	0.021,276,6	6.855,654,600,4	97	0.010,309,3	9.848,857,801,8
48	0.020,833,3	6.928,203,230,3	98	0.010,204,1	9.899,494,936,6
49	0.020,408,2	7.000,000,000,0	99	0.010,101,0	9.949,874,371,1
50	0.02	7.071,067,811,9	100	0.01	10.000,000,000,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
101	0.009,900,9	10.049,875,621,1	151	0.006,622,5	12.288,205,727,4
102	0.009,803,9	10.099,504,938,4	152	0.006,578,9	12.328,828,005,9
103	0.009,708,7	10.148,891,565,1	153	0.006,535,9	12.369,316,876,9
104	0.009,615,4	10.198,039,027,2	154	0.006,493,5	12.409,673,646,0
105	0.009,523,8	10.246,950,766,0	155	0.006,451,6	12.449,899,598,0
106	0.009,434,0	10.295,630,141,0	156	0.006,410,3	12.489,995,996,8
107	0.009,345,8	10.344,080,432,8	157	0.006,369,4	12.529,964,086,1
108	0.009,259,2	10.392,304,845,4	158	0.006,329,1	12.569,805,090,0
109	0.009,174,3	10.440,306,508,9	159	0.006,289,3	12.609,520,212,9
110	0.009,090,9	10.488,088,481,7	160	0.006,25	12.649,110,640,7
111	0.009,009,0	10.535,653,752,9	161	0.006,211,2	12.688,577,540,4
112	0.008,928,6	10.583,005,244,3	162	0.006,172,8	12.727,922,061,4
113	0.008,849,6	10.630,145,812,7	163	0.006,135,0	12.767,145,334,8
114	0.008,771,9	10.677,078,252,0	164	0.006,097,5	12.806,248,474,9
115	0.008,695,7	10.723,805,294,8	165	0.006,060,6	12.845,232,578,7
116	0.008,620,7	10.770,329,614,3	166	0.006,024,1	12.884,098,726,7
117	0.008,547,0	10.816,653,826,4	167	0.005,988,0	12.922,847,983,3
118	0.008,474,5	10.862,780,491,2	168	0.005,952,4	12.961,481,396,8
119	0.008,403,4	10.908,712,114,6	169	0.005,917,2	13.000,000,000,0
120	0.008,333,3	10.954,451,150,1	170	0.005,882,4	13.038,404,810,4
121	0.008,264,5	11.000,000,000,0	171	0.005,848,0	13.076,696,830,6
122	0.008,196,7	11.045,361,017,2	172	0.005,814,0	13.114,877,048,6
123	0.008,130,0	11.090,536,506,4	173	0.005,780,3	13.152,946,438,0
124	0.008,064,5	11.135,528,725,7	174	0.005,747,1	13.190,905,958,3
125	0.008,	11.180,339,887,5	175	0.005,714,3	13.228,756,555,3
126	0.007,936,5	11.224,972,160,3	176	0.005,681,8	13.266,499,161,4
127	0.007,874,0	11.269,427,669,6	177	0.005,649,7	13.304,134,695,7
128	0.007,812,5	11.313,708,499,0	178	0.005,618,0	13.341,664,064,1
129	0.007,751,9	11.357,816,691,6	179	0.005,586,6	13.379,088,160,3
130	0.007,692,3	11.401,754,251,0	180	0.005,555,5	13.416,407,865,0
131	0.007,633,6	11.445,523,142,3	181	0.005,524,9	13.453,624,047,1
132	0.007,575,7	11.489,125,293,1	182	0.005,494,5	13.490,737,563,2
133	0.007,518,8	11.532,562,594,7	183	0.005,464,5	13.527,749,258,5
134	0.007,462,7	11.575,836,902,8	184	0.005,434,8	13.564,659,966,3
135	0.007,407,4	11.618,950,038,6	185	0.005,405,4	13.601,470,508,7
136	0.007,352,9	11.661,903,789,7	186	0.005,376,3	13.638,181,697,0
137	0.007,299,3	11.704,699,911,1	187	0.005,347,6	13.674,794,331,2
138	0.007,246,4	11.747,344,380,8	188	0.005,319,1	13.711,309,200,8
139	0.007,194,2	11.789,826,122,6	189	0.005,291,0	13.747,727,084,9
140	0.007,142,9	11.832,159,566,2	190	0.005,263,2	13.784,048,752,1
141	0.007,092,2	11.874,342,087,0	191	0.005,235,6	13.820,274,961,1
142	0.007,042,3	11.916,375,287,8	192	0.005,208,3	13.856,406,460,6
143	0.006,993,0	11.958,260,743,1	193	0.005,181,3	13.892,443,989,4
144	0.006,944,4	12.000,000,000,0	194	0.005,154,6	13.928,388,277,2
145	0.006,896,6	12.041,594,578,8	195	0.005,128,2	13.964,240,043,8
146	0.006,849,3	12.083,045,973,6	196	0.005,102,0	14.000,000,000,0
147	0.006,802,7	12.124,355,653,0	197	0.065,076,1	14.035,668,844,1
148	0.006,756,7	12.165,525,060,6	198	0.005,050,5	14.071,247,279,5
149	0.006,711,4	12.206,555,615,3	199	0.005,025,1	14.106,735,979,7
150	0.006,666,6	12.247,448,713,9	200	0.005,	14.142,135,623,7

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
201	0.004,975,1	14.177,446,878,8	251	0.003,984,1	15.842,979,517,8
202	0.004,950,4	14.212,670,403,6	252	0.003,968,3	15.874,507,866,4
203	0.004,926,1	14.247,806,848,8	253	0.003,952,6	15.905,973,720,6
204	0.004,902,0	14.282,856,857,1	254	0.003,937,0	15.937,377,450,5
205	0.004,875,0	14.317,821,063,3	255	0.003,921,6	15.968,719,422,7
206	0.004,854,4	14.352,700,094,4	256	0.003,906,3	16.000,000,000,0
207	0.004,830,9	14.387,494,569,9	257	0.003,891,1	16.031,219,541,9
208	0.004,807,7	14.422,205,101,9	258	0.003,876,0	16.062,378,404,2
209	0.004,784,7	14.456,832,294,8	259	0.003,861,0	16.093,476,939,4
210	0.004,761,9	14.491,376,746,2	260	0.003,846,2	16.124,515,496,6
211	0.004,739,3	14.525,839,046,3	261	0.003,831,4	16.155,494,421,4
212	0.004,717,0	14.560,219,778,6	262	0.003,816,8	16.186,414,056,2
213	0.004,694,8	14.594,519,519,3	263	0.003,802,3	16.217,274,740,2
214	0.004,672,9	14.628,738,838,3	264	0.003,787,8	16.248,076,809,2
215	0.004,651,2	14.662,878,298,6	265	0.003,773,6	16.278,820,596,1
216	0.004,629,6	14.696,938,456,7	266	0.003,759,4	16.309,506,430,3
217	0.004,608,3	14.730,919,862,7	267	0.003,745,3	16.340,134,638,4
218	0.004,587,2	14.764,823,060,2	268	0.003,731,3	16.370,705,543,7
219	0.004,566,2	14.798,648,586,9	269	0.003,717,5	16.401,219,466,9
220	0.004,545,4	14.832,396,974,2	270	0.003,703,7	16.431,676,725,2
221	0.004,524,9	14.866,068,747,3	271	0.003,690,0	16.462,077,633,2
222	0.004,504,5	14.899,664,425,8	272	0.003,676,5	16.492,422,502,5
223	0.004,484,3	14.933,184,523,1	273	0.003,663,0	16.522,711,641,9
224	0.004,464,3	14.966,629,547,1	274	0.003,649,6	16.552,945,356,9
225	0.004,444,4	15.000,000,000,0	275	0.003,636,3	16.583,123,951,8
226	0.004,424,8	15.033,296,378,4	276	0.003,623,2	16.613,247,725,8
227	0.004,405,3	15.066,519,173,3	277	0.003,610,1	16.643,316,977,1
228	0.004,386,0	15.099,668,870,5	278	0.003,597,1	16.673,332,000,5
229	0.004,366,8	15.132,745,950,4	279	0.003,584,2	16.703,293,088,5
230	0.004,347,8	15.165,750,888,1	280	0.003,571,4	16.733,200,530,7
231	0.004,329,0	15.198,684,153,6	281	0.003,558,7	16.763,054,614,2
232	0.004,310,3	15.231,546,211,7	282	0.003,546,1	16.792,855,623,7
233	0.004,291,8	15.264,337,522,5	283	0.003,533,6	16.822,603,841,3
234	0.004,273,5	15.297,058,540,8	284	0.003,521,1	16.852,299,546,4
235	0.004,255,3	15.329,709,716,8	285	0.003,508,8	16.881,943,016,1
236	0.904,237,3	15.362,291,495,7	286	0.003,496,5	16.911,534,525,3
237	0.004,219,4	15.394,804,318,3	287	0.003,484,3	16.941,074,346,1
238	0.004,201,7	15.427,248,620,9	288	0.003,472,2	16.970,562,748,5
239	0.004,184,1	15.459,624,833,7	289	0.003,460,2	17.000,000,000,0
240	0.004,166,6	15.491,933,384,8	290	0.003,448,3	17.029,386,365,9
241	0.004,149,4	15.524,174,696,3	291	0.003,436,4	17.058,722,109,2
242	0.004,132,2	15.556,349,186,1	292	0.003,424,6	17.088,007,490,6
243	0.004,115,2	15.588.457,268,1	293	0.003,413,0	17.117,242,768,6
244	0.004,098,4	15.620,499,351,8	294	0.003,401,4	17.146,428,199,5
245	0.004,081,6	15.652,475,842,5	295	0.003,389,8	17.175,564,037,3
246	0.004,065	15.684,387,141,4	296	0.003,378,3	17.204,650,534,1
247	0.004,048,6	15.716,233,645,5	297	0.003,367,0	17.233,687,939,6
248	0.004,032,3	15.748,015,748,0	298	0.003,355,7	17.262,676,501,6
249	0.004,016,1	15.779,733,838,1	299	0.003,344,5	17.291,616,465,8
250	0.004,	15.811,388,300,8	300	0.003,333,3	17.320,508,075,7

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
301	0.003,322,3	17.349,351,572,9	351	0.002,849,0	18.734,993,995,2
302	0.003,311,3	17.378,147,196,9	352	0.002,840,9	18.761,663,039,3
303	0.003,300,3	17.406,895,185,5	353	0.002,832,9	18.788,294,228,1
304	0.003,289,5	17.435,595,774,2	354	0.002,824,8	18.814,887,722,2
305	0.003,278,7	17.464,249,196,6	355	0.002,816,9	18.841,443,681,4
306	0.003,268,0	17.492,855,684,5	356	0.002,809,0	18.867,962,264,1
307	0.003,257,3	17.521,415,467,9	357	0.002,801,1	18.894,443,627,7
308	0.003,246,8	17.549,928,774,8	358	0.002,793,3	18.920,887,928,4
309	0.003,236,2	17.578,395,831,2	359	0.002,785,5	18.947,295,321,5
310	0.003,225,8	17.606,816,861,7	360	0.002,777,7	18.973,665,961,0
311	0.003,215,4	17.635,192,088,5	361	0.002,770,1	19.000,000,000,0
312	0.003,205,1	17.663,521,732,7	362	0.002,762,4	19.026,297,590,4
313	0.003,194,9	17.691,806,013,0	363	0.002,754,8	19.052,558,883,3
314	0.003,184,7	17.720,045,146,7	364	0.002,747,3	19.078,784,028,3
315	0.003,174,6	17.748,239,349,3	365	0.002,739,7	19.104,973,174,5
316	0.003,164,6	17.776,388,834,6	366	0.002,732,2	19.131,126,469,7
317	0.003,154,6	17.804,493,814,8	367	0.002,724,8	19.157,244,060,7
318	0.003,144,7	17.832,554,500,1	368	0.002,717,4	19.183,326,093,3
319	0.003,134,8	17.860,571,099,5	369	0.002,710,0	19.209,372,712,3
320	0.003,125,	17.888,543,820,0	370	0.002,702,7	19.235,384,061,7
321	0.003,115,3	17.916,472,867,2	371	0.002,695,4	19.261,360,284,3
322	0.003,105,6	17.944,358,444,9	372	0.002,688,2	19.287,301,522,0
323	0.003,096,0	17.972,200,755,6	373	0.002,681,0	19.313,207,915,8
324	0.003,086,4	18.000,000,000,0	374	0.002,673,8	19.339,053,751,4
325	0.003,076,9	18.027,756,377,3	375	0.002,666,6	19.364,916,731,0
326	0.003,067,5	18.055,470,085,3	376	0.002,659,6	19.390,719,429,7
327	0.003,058,1	18.083,141,320,0	377	0.002,652,5	19.416,487,838,9
328	0.003,048,8	18.110,770,276,3	378	0.002,645,5	19.442,222,095,2
329	0.003,039,5	18.138,357,147,2	379	0.002,638,5	19.467,922,333,9
330	0.003,030,3	18.165,902,124,6	380	0.002,631,6	19.493,588,689,6
331	0.003,021,1	18.193,405,398 7	381	0.002,624,7	19.519,221,295,9
332	0.003,012,0	18.220,867,158,3	382	0.002,617,8	19.544,820,285,7
333	0.003,003,0	18.248,287,590,9	383	0.002,611,0	19.570,385,790,8
334	0.002,994,0	18.275,666,882,5	384	0.002,604,2	19.595,917,942,3
335	0 002,985,1	18.303,005,217,7	385	0.002,597,4	19.621,416,870,3
336	0.002,976,2	18.330,302,779,8	386	0.002,590,7	19.646,882,704,4
337	0.002,967,4	18.357,559,750,7	387	0.002,584,0	19.672,315,572,9
338	0.002,958,6	18.384,776,310,9	388	0.002,577,3	19.697,715,603,6
339	0.002,949,9	18.411,952,639,5	389	0.002,570,7	19.723,082,923,1
340	0.002,941,2	18.439,088,914,6	390	0.002,564,1	19.748,417,658,1
341	0.002,932,6	18.466,185,312,6	391	0.002,557,5	19.773,719,933,3
342	0.002,924,0	18.493,242,008,9	392	0.002,551,0	19.798,989,873,2
343	0.002,915,5	18 520,259,177,5	393	0.002,544,5	19.824,227,601,6
344	0.002,907,0	18.547,236,991,0	394	0.002,538,1	19.849,433,241,3
345	0.002,898,6	18.574,175,621,0	395	0.002,531,6	19.874,606,914,4
346	0.002,890,2	18.601,075,237,7	396	0.002,525,2	19.899,748,742,1
347	0.002,881,8	18.627,936,010,2	397	0.002,518,9	19.924,858,845,2
348	0.002,873,6	18.654,758,106,2	398	0.002,512,6	19.949,937,343,3
349	0.002,865,3	18.681,541,692,3	399	0.002,506,3	19.974,984,355,4
350	0.002,857,1	18.708,286,933,9	400	0.002,5	20.000,000,000,0

No.	Reciprocal	Square Root.
401	0.002,493,8	20.024,984,394,5
402	0.002,487,6	20.049,937,655,8
403	0.002,481,4	20.074,859,899,9
404	0.002,475,2	20.099,751,242,2
405	0.002,469,1	20.124,611,797,5
406	0.002,463,1	20.149,441,679,6
407	0.002,457,0	20.174,241,001,8
408	0.002,451,0	20.199,009,876,7
409	0.002,445,0	20.223,748,416,2
410	0.002,439,0	20.248,456,731,3
411	0.002,433,1	20.273,134,932,7
412	0.002,427,2	20.297,783,130,2
413	0.002,421,3	20.322,401,432,9
414	0.002,415,5	20.346,989,949,4
415	0.002,409,6	20.371,548,787,5
416	0.002,403,8	20.396,078,054,4
417	0.002,398,1	20.420,577,856,7
418	0.002,392,3	20.445,048,300,3
419	0.002,386,6	20.469,489,490,5
420	0.002,381,0	20.493,901,531,9
421	0.002,375,3	20.518,284,528,7
422	0.002,369,7	20.542,638,584,2
423	0.002,364,1	20.566,963,801,2
424	0.002,358,5	20.591,260,282,0
425	0.002,352,9	20.615,528,128,1
426	0.002,347,4	20.639,767,440,6
427	0.002,341,9	20.663,978,319,8
428	0.002,336,4	20.688,160,865,6
429	0.002,331,0	20.712,315,177,2
430	0.002,325,6	20.736,441,353,3
431	0.002,320,2	20.760,539,492,0
432	0.002,314,8	20.784,609,690,8
433	0.002,309,5	20.808,652,046,7
434	0.002,304,2	20.832,666,656,0
435	0.002,298,9	20.856,653,614,6
436	0.002,293,6	20.880,613,017,8
437	0.002,288,3	20.904,544,960,4
438	0.002,283,1	20.928,449,536,5
439	0.002,277,9	20.952,326,839,8
440	0.002,272,7	20.976,176,963,4
441	0.002,267,6	21.000,000,000,0
442	0.002,262,4	21.023,796,041,6
443	0.002,257,3	21.047,565,179,8
444	0.002,252,2	21.071,307,505,7
445	0.002,247,2	21.095,023,109,7
446	0.002,242,2	21.118,712,081,9
447	0.002,237,1	21.142,374,511,9
448	0.002,232,1	21.166,010,488,5
449	0.002,227,2	21.189,620,100,4
450	0.002,222,2	21.213,203,435,6

No.	Reciprocal	Square Root.
451	0.002,217,3	21.236,760,581,6
452	0.002,212,4	21.260,291,625,5
453	0.002,207,5	21.283,796,653,8
454	0.002,202,6	21.307,275,752,7
455	0.002,197,8	21.330,729,007,7
456	0.002,193,0	21.354,156,504,1
457	0.002,188,2	21.377,558,326,4
458	0.002,183,4	21.400,934,559,0
459	0.002,178,6	21.424,285,285,6
460	0.002,173,9	21.447,610,589,5
461	0.002,169,2	21.470,910,553,6
462	0.002,164,5	21.494,185,257,9
463	0.002,159,8	21.517,434,791,4
464	0.002,155,2	21.540,659,228,5
465	0.002,150,5	21.563,858,652,8
466	0.002,145,9	21.587,033,144,9
467	0.002,141,3	21.610,182,785,0
468	0.002,136,8	21.633,307,652,8
469	0.002,132,2	21.656,407,827,7
470	0.002,127,7	21.679,483,388,7
471	0.002,123,1	21.702,534,414,2
472	0.002,118,7	21.725,560,982,4
473	0.002,114,2	21.748,563,170,9
474	0.092,109,7	21.771,541,057,1
475	0.002,105,3	21.794,494,717,7
476	0.002,100,8	21.817,424,229,3
477	0.002,096,4	21.840,329,667,8
478	0.002,092,1	21.863,211,109,1
479	0.002,087,7	21.886,068,628,2
480	0.002,083,3	21.908,902,300,2
481	0.002,079,0	21.931,712,199,5
482	0.002,074,7	21.954,498,402,4
483	0.002,070,4	21.977,260,975,8
484	0.002,066,1	22.000,000,000,0
485	0.002,061,9	22.022,715,545,5
486	0.002,057,6	22.045,407,685,0
487	0.002,053,4	22.068,076,490,7
488	0.002,049,2	22.090,722,034,4
489	0.002,045,0	22.113,344,387,5
490	0.002,040,8	22.135,943,621,2
491	0.002,036,7	22.158,519,806,2
492	0.002,032,5	22.181,073,012,8
493	0.002,028,4	22.203,003,311,2
494	0.002,024,3	22.226,110,770,9
495	0.002,020,2	22.248,595,461,3
496	0.002,016,2	22.271,057,451,3
497	0.002,012,1	22.293,496,809,6
498	0.002,008,0	22.315,913,604,4
499	0.002,004,0	22.338,307,903,9
500	0.002.	22.360,679,775,0

No.	Reciprocal	Square Root.
501	0.001,996,0	22.383,029,285,6
502	0.001,992,0	22.405,356,502,4
503	0.001,988,1	22.427,661,492,0
504	0.001,984,1	22.449,944,320,6
505	0.001,980,1	22.472,205,054,2
506	0.001,976,3	22.494,443,758,4
507	0.001,972,4	22.516,660,498,4
508	0.001,968,5	22.538,855,339,2
509	0.001,964,6	22.561,028,345,4
510	0.001,960,8	22.583,179,581,3
511	0.001,956,9	22.605,309,110,9
512	0.001,953,1	22.627,416,998,0
513	0.001,949,3	22.649,503,305,8
514	0.001,945,5	22.671,568,097,5
515	0.001,941,7	22.693,611,435,8
516	0.001,938,0	22.715,633,383,2
517	0.001,934,2	22.737,634,001,8
518	0.001,930,5	22.759,613,353,5
519	0.001,926,8	22.781,571,499,8
520	0.001,923,1	22.803,508,502,0
521	0.001,919,4	22.825,424,421,0
522	0.001,915,7	22.847,319,317,6
523	0.001,912,0	22.869,193,252,1
524	0.001,908,4	22.891,046,284,5
525	0.001,904,8	22.912,878,474,8
526	0.001,901,1	22.934,689,882,4
527	0.001,897,5	22.956,480,566,5
528	0.001,893,9	22.978,250,586,2
529	0.001,890,4	23.000,000,000,0
530	0.001,886,8	23.021,728,866,4
531	0.001,883,2	23.043,437,243,6
532	0.001,879,7	23.065,125,189,3
533	0.001,876,2	23.086,792,761,2
534	0.001,872,7	23.108,440,016,6
535	0.001,869,2	23.130,067,012,4
536	0.001,865,7	23.151,673,805,6
537	0.001,862,2	23.173,260,452,5
538	0.001,858,7	23.194,827,009,5
539	0.001,855,3	23.216,373,532,5
540	0.001,851,8	23.237,900,077,2
541	0.001,848,4	23.259,406,699,2
542	0.001,845,0	23.280,893,453,6
543	0.001,841,6	23.302,360,395,5
544	0.001,838,2	23.323,807,579,4
545	0.001,834,9	23.345,235,059,9
546	0.001,831,5	23.366,642,891,1
547	0.001,828,2	23.388,031,127,1
548	0.001,824,8	23.409,399,821,4
549	0.001,821,5	23.430,749,027,7
550	0.001,818,1	23.452,078,799,1
551	0.001,814,9	23.473,389,188,6
552	0.001,811,6	23.494,680,248,9
553	0.001,808,3	23.515,952,032,6
554	0.001,805,1	23.537,204,591,9
555	0.001,801,8	23.558,437,978,8
556	0.001,798,6	23.579,652,245,1
557	0.001,795,3	23.600,847,442,4
558	0.001,792,1	23.622,023,622,0
559	0.001,788,9	23.643,180,835,1
560	0.001,785,7	23.664,319,132,4
561	0.001,782,5	23.685,438,564,7
562	0.001,779,4	23.706,539,182,3
563	0.001,776,2	23.727,621,035,4
564	0.001,773,0	23.748,684,174,1
565	0.001,769,9	23.769,728,648,0
566	0.001,766,8	23.790,754,506,7
567	0.001,763,7	23.811,761,799,6
568	0.001,760,6	23.832,750,575,6
569	0.001,757,5	23.853,720,883,8
570	0.001,754,4	23.874,672,772,6
571	0.001,751,3	23.895,606,290,7
572	0.001,748,3	23.916,521,486,2
573	0.001,745,2	23.937,418,407,2
574	0.001,742,2	23.958,297,101,4
575	0.001,739,1	23.979,157,616,6
576	0.001,736,1	24.000,000,000,0
577	0.001,733,1	24.020,824,298,9
578	0.001,730,1	24.041,630,560,3
579	0.001,727,1	24.062,418,831,0
580	0.001,724,1	24.083,168,396,2
581	0.001,721,2	24.103,941,586,4
582	0.001,718,2	24.124,676,163,6
583	0.001,715,3	24.145,392,935,3
584	0.001,712,3	24.166,091,947,2
585	0.001,709,4	24.186,773,244,9
586	0.001,706,5	24.207,436,873,6
587	0.001,703,6	24.228,082,879,2
588	0.001,700,7	24.248,711,306,0
589	0.001,697,8	24.269,322,199,0
590	0.001,694,9	24.289,915,603,0
591	0.001,692,0	24.310,491,562,3
592	0.001,689,1	24.331,050,121,2
593	0.001,686,3	24.351,591,323,8
594	0.001,683,5	24.372,115,213,9
595	0.001,680,7	24.392,621,835,3
596	0.001,677,9	24.413,111,231,5
597	0.001,675,0	24.433,583,445,7
598	0.001,672,2	24.454,038,521,3
599	0.001,669,4	24.474,476,501,0
600	0.001,666,6	24.494,897,427,8

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
601	0.001,663,9	24.515,301,344,3	651	0.001,536,1	25.514,701,644,3
602	0.001,661,1	24.535,688,292,8	652	0.001,533,7	25.534,290,669,6
603	0.001,658,4	24.556,058,315,6	653	0.001,531,4	25.553,864,678,4
604	0.001,655,6	24.576,411,454,9	654	0.001,529,1	25.573,423,705,1
605	0.001,652,9	24.596,747,752,5	655	0.001,526,7	25.592,967,784,1
606	0.001,650,1	24.617,067,250,2	656	0.001,524,4	25.612,496,949,7
607	0.001,647,4	24.637,369,989,5	657	0.001,522,1	25.632,011,236,0
608	0.001,644,7	24.657,656,011,9	658	0.001,519,8	25.651,510,676,8
609	0.001,642,0	24.677,925,358,5	659	0.001,517,5	25.670,995,306,0
610	0.001,639,3	24.698,178,070,5	660	0.001,515,1	25.690,465,157,3
611	0.001,636,7	24.718,414,188,6	661	0.001,512,9	25.709,920,264,4
612	0.001,634,0	24.738,633,753,7	662	0.001,510,6	25.729,360,660,5
613	0.001,631,3	24.758,836,806,3	663	0.001,508,3	25.748,786,379,2
614	0.001,628,7	24.779,023,386,7	664	0.001,506,0	25.768,197,453,5
615	0.001,626,0	24.799,193,535,3	665	0.001,503,8	25.787,593,916,5
616	0.001,623,4	24.819,347,292,0	666	0.001,501,5	25.806,975,801,1
617	0.001,620,7	24.839,484,696,7	667	0.001,499,3	25.826,343,140,3
618	0.001,618,1	24.859,605,789,3	668	0.001,497,0	25.845,695,966,6
619	0.001,615,5	24.879,710,609,2	669	0.001,494,8	25.865,034,312,8
620	0.001,612,9	24.899,799,196,0	670	0.001,492,5	25.884,358,211,1
621	0.001,610,3	24.919,871,588,8	671	0.001,490,3	25.903,667,694,0
622	0.001,607,7	24.939,927,826,7	672	0.001,488,1	25.922,962,793,6
623	0.001,605,1	24.959,967,948,7	673	0.001,485,9	25.942,243,542,1
624	0.001,602,6	24.979,991,993,6	674	0.001,483,7	25.961,509,971,5
625	0.001,6	25.000,000,000,0	675	0.001,481,4	25.980,762,113,5
626	0.001,597,4	25.019,992,006,4	676	0.001,479,3	26.000,000,000,0
627	0.001,594,9	25.039,968,051,1	677	0.001,477,1	26.019,223,662,5
628	0.001,592,4	25.059,928,172,3	678	0.001,474,9	26.038,433,132,6
629	0.001,589,8	25.079,872,408,0	679	0.001,472,8	26.057,628.441,6
630	0.001,587,3	25.099,800,796,0	680	0.001,470,6	26.076,809,620,8
631	0.001,584,8	25.119,713,374,2	681	0.001,468,4	26.095,976,701,4
632	0.001,582,3	25.139,610,180,0	682	0.001,466,3	26.115,129,714,4
633	0.001,579,8	25.159,491,250,8	683	0.001,464,1	26.134,268,690,7
634	0.001,577,3	25.179,356,620,1	684	0.001,462,0	26.153,393,661,2
635	0.001,574,8	25.199,206,336,7	685	0.001,459,9	26.172,504,656,6
636	0.001,572,3	25.219,040,425,8	686	0.001,457,7	26.191,601,707,4
637	0.001,569,9	25.238,858,928,2	687	0.001,455,6	26.210,684,844,2
638	0.001,567,4	25.258,661,880,6	688	0.001,453,5	26.229,754,097,2
639	0.001,564,9	25.278,449,319,5	689	0.001,451,4	26.248,809,496,8
640	0.001,562,5	25.298,221,281,3	690	0.001,449,3	26.267,851,073,1
641	0.001,560,1	25.317,977,802,3	691	0.001,447,2	26.286,878,856,2
642	0.001,557,6	25.337,718,918,6	692	0.001,445,1	26.305,892,875,9
643	0.001,555,2	25.357,444,666,2	693	0.001,443,0	26.324,893,162,2
644	0.001,552,8	25.377,155,080,9	694	0.001,440,9	26.343,879,744,6
645	0.001,550,4	25.396,850,198,4	695	0.001,438,8	26.362,852,652,9
646	0.001,548,0	25.416,530,054,3	696	0.001,436,8	26.381,811,916,5
647	0.001,545,6	25.436,194,684,0	697	0.001,434,7	26.400,757,564,9
648	0.001,543,2	25.455,844,122,7	698	0.001,432,7	26.419,689,627,2
649	0.001,540,8	25.475,478,405,7	699	0.001,430,6	26.438,608,132,8
650	0.001,538,5	25.495,097,568,0	700	0.001,428,6	26.457,513,110,6

No.	Reciprocal	Square Root.
701	0.001,426,5	26.476,404,589,7
702	0.001,424,5	26.495,282,599,0
703	0.001,422,5	26.514,147,167,1
704	0.001,420,5	26.532,098,322,8
705	0.001,418,4	26.551,836,094,7
706	0.001,416,4	26.570,660,511,2
707	0.001,414,4	26.589,471,600,6
708	0.001,412,4	26.608,269,391,3
709	0.001,410,4	26.627,053,911,4
710	0.001,408,5	26.645,825,188,9
711	0.001,406,5	26.664,583,251,9
712	0.001,404,5	26.683,328,128,3
713	0.001,402,5	26.702,059,845,6
714	0.001,400,6	26.720,778,431,8
715	0.001,398,6	26.739,483,914,2
716	0.001,396,6	26.758,176,320,5
717	0.001,394,7	26.776,855,678,0
718	0.001,392,8	26.795,522,013,9
719	0.001,390,8	26.814,175,355,6
720	0.001,388,8	26.832,815,730,0
721	0.001,387,0	26.851,443,164,2
722	0.001,385,0	26.870,057,685,1
723	0.001,383,1	26.888,659,319,5
724	0.001,381,2	26.907,248,094,1
725	0.001,379,3	26.925,824,035,7
726	0.001,377,4	26.944,387,170,6
727	0.001,375,5	26.962,937,525,4
728	0.001,373,6	26.981,475,126,5
729	0.001,371,7	27.000,000,000,0
730	0.001,369,9	27.018,512,172,2
731	0.001,368,0	27.037,011,669,2
732	0.001,366,1	27.055,498,516,9
733	0.001,364,3	27.073,972,741,4
734	0.001,362,4	27.092,434,368,3
735	0.001,360,5	27.110,883,423,5
736	0.001,358,7	27.129,319,932,5
737	0.001,356,9	27.147,743,921,0
738	0.001,355,0	27.166,155,414,4
739	0.001,353,2	27.184,554,438,1
740	0.001,351,3	27.202,941,017,5
741	0.001,349,5	27.221,315,177,6
742	0.001,347,7	27.239,676,943,8
743	0.001,345,9	27.258,026,340,9
744	0.001,344,1	27.276,363,394,0
745	0.001,342,3	27.294,688,127,9
746	0.001,340,5	27.313,000,567,5
747	0.001,338,7	27.331,300,737,4
748	0.001,336,9	27.349,588,662,4
749	0.001,335,1	27.367,864,366,8
750	0.001,333,3	27.386,127,875,3

No.	Reciprocal	Square Root.
751	0.001,331,6	27.404,379,212,1
752	0.001,329,8	27.422,618,401,6
753	0.001,328,0	27.440,845,468,0
754	0.001,326,3	27.459,060,435,5
755	0.001,324,5	27.477,263,328,1
756	0.001,322,8	27.495,454,169,7
757	0.001,321,0	27.513,632,984,4
758	0.001,319,3	27.531,799,795,9
759	0.001,3 7,5	27.549,954,627,9
760	0.001,315,8	27.568,097,504,2
761	0.001,314,1	27.586,228,448,3
762	0 001,312,3	27.604,347,483,7
763	0.001,310,6	27.622,454,633,9
764	0.001,308,9	27.640,549,922,2
765	0.001,307,2	27.658,633,371,9
766	0.001,305,5	27.676,705,006,2
767	0.001,303,8	27.694,764,848,3
768	0.001,302,1	27.712,812,921,1
769	0.001,300,4	27.730,849,247,7
770	0.001,298,7	27.748,873,851,0
771	0.001,297,0	27.766,886,753,8
772	0.001,295,3	27.784,887,978,9
773	0.001,293,7	27.802,877,548,9
774	0.001,292,0	27.820,855,486,5
775	0.001,290,3	27.838,821,814,2
776	0.001,288,7	27.856,776,554,4
777	0.001,287,0	27.874,719,729,5
778	0.001,285,3	27.892,651,362,0
779	0.001,283,7	27.910,571,473,9
780	0.001,282,1	27.928,480,087,5
781	0.001,280,4	27.946,377,225,0
782	0.001,278,8	27.964,262,908,2
783	0.001,277,1	27.982,137,159,3
784	0.001,275,5	28.000,000,000,0
785	0.001,273,9	28.017,851,452,2
786	0.001,272,3	28.035,691,537,8
787	0.001,270,6	28.053,520,278,2
788	0.001,269,0	28.071,337,688,1
789	0.001,267,4	28.089,143,810,4
790	0.001,265,8	28.106,938,645,1
791	0.001,264,2	28.124,722,220,9
792	0.001,262,6	28.142,494,558,9
793	0.001,261,0	28.160,255,680,7
794	0.001,259,4	28.178,005,607,2
795	0.001,257,9	28.195,744,359,7
796	0.001,256,3	28.213,471,959,3
797	0.001,254,7	28.231,188,427,0
798	0.001,253,1	28.248,893,783,7
799	0.001,251,6	28.266,588,050,2
800	0.001,25	28.284,271,247,5

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
801	0.001,248,4	28.301,943,396,2	851	0.001,175,1	29.171,904,291,6
802	0.001,246,9	28.319,604,517,0	852	0.001,173,7	29.189,039,038,7
803	0.001,245,3	28.337,254,630,6	853	0.001,172,3	29.206,163,733,0
804	0.001,243,8	28.354,893,757,5	854	0.001,171,0	29.223,278,392,4
805	0.001,242,2	28.372,521,918,2	855	0.001,169,6	29.240,383,034,4
806	0.001,240,7	28.390,139,133,2	856	0.001,168,2	29.257,477,676,7
807	0.001,239,2	28.407,745,422,7	857	0.001,166,9	29.274,562,336,6
808	0.001,237,6	28.425,340,807,1	858	0.001,165,5	29.291,637,031,8
809	0.001,236,1	28.442,925,306,7	859	0.001,164,1	29.308,701,779,5
810	0.001,234,6	28.460,498,941,5	860	0.001,162,8	29.325,756,597,2
811	0.001,233,0	28.478,061,731,8	861	0.001,161,4	29.342,801,502,2
812	0.001,231,5	28.495,613,697,6	862	0.001,160,1	29.359,836,511,8
813	0.001,230,0	28.513,154,858,8	863	0.001,158,7	29.376,861,643,1
814	0.001,228,5	28.530,685,235,4	864	0.001,157,4	29.393,876,913,4
815	0.001,227,0	28.548,204,847,2	865	0.001,156,1	29.410,882,339,7
816	0.001,225,5	28.565,713,714,2	866	0.001,154,7	29.427,877,939,1
817	0.001,224,0	28.583,211,855,9	867	0.001,153,4	29.444,863,728,7
818	0.001,222,5	28.600,699,292,2	868	0.001,152,1	29.461,839,725,3
819	0.001,221,0	28.618,176,042,5	869	0.001,150,7	29.478,805,946,0
820	0.001,219,5	28.635,642,126,6	870	0.001,149,4	29.495,762,407,5
821	0.001,218,0	28.653,097,563,8	871	0.001,148,1	29.512,709,126,7
822	0.001,216,5	28.670,542,373,7	872	0.001,146,8	29.529,646,120,5
823	0.001,215,1	28.687,976,575,6	873	0.001,145,5	29.546,573,405,4
824	0.001,213,6	28.705,400,188,8	874	0.001,144,2	29.563,490,998,2
825	0.001,212,1	28.722,813,232,7	875	0.001,142,9	29.580,398,915,5
826	0.001,210,6	28.740,215,726,4	876	0.001,141,6	29.597,297,173,9
827	0.001,209,2	28.757,607,689,1	877	0.001,140,3	29.614,185,789,9
828	0.001,207,7	28.774,989,139,9	878	0.001,139,0	29.631,064,780,1
829	0.001,206,3	28.792,360,097,8	879	0.001,137,7	29.647,932,474,3
830	0.001,204,8	28.809,720,581,8	880	0.001,136,3	29.664,793,948,4
831	0.001,203,4	28.827,070,610,8	881	0.001,135,1	29.681,644,159,3
832	0.001,201,9	28.844,410,203,7	882	0.001,133,8	29.698,484,809,8
833	0.001,200,5	28.861,739,379,3	883	0.001,132,5	29.715,315,916,2
834	0.001,199,0	28.879,058,156,4	884	0.001,131,2	29.732,137,494,6
835	0.001,197,6	28.896,366,553,6	885	0.001,129,9	29.748,949,561,3
836	0.001,196,2	28.913,664,589,6	886	0.001,128,7	29.765,752,132,3
837	0.001,194,7	28.930,952,283,0	887	0.001,127,4	29.782,545,223,7
838	0.001,193,3	28.948,229,652,3	888	0.001,126,1	29.799,328,851,5
839	0.001,191,9	28.965,496,715,9	889	0.001,124,9	29.816,103,031,8
840	0.001,190,5	28.982,753,492,4	890	0.001,123,6	29.832,867,780,4
841	0.001,189,1	29.000,000,000,0	891	0.001,122,3	29.849,623,113,2
842	0.001,187,6	29.017,236,257,1	892	0.001,121,1	29.866,369,046,1
843	0.001,186,2	29.034,462,281,9	893	0.001,119,8	29.883,105,595,0
844	0.001,184,8	29.051,678,092,7	894	0.001,118,6	29.899,832,775,5
845	0.001,183,4	29.068,883,707,5	895	0.001,117,3	29.916,550,603,3
846	0.001,182,0	29.086,079,144,5	896	0.001,116,1	29.933,259,094,2
847	0.001,180,6	29.103,264,421,7	897	0.001,114,8	29.949,958,263,7
848	0.001,179,2	29.120,439,557,1	898	0.001,113,6	29.966,648,127,5
849	0.001,177,9	29.137,604,568,7	899	0.001,112,3	29.983,328,701,1
850	0.001,176,5	29.154,759,474,2	900	0.001,111,1	30.000,000,000,0

No.	Reciprocal	Square Root.	No.	Reciprocal	Square Root.
901	0.001,109,9	30.016,662,039,6	951	0.001,051,5	30.838,287,890,2
902	0.001,108,6	30.033,314,835,4	952	0.001,050,4	30.854,497,241,7
903	0.001,107,4	30.049,958,402,6	953	0.001,049,3	30.870,698,080,9
904	0.001,106,2	30.066,592,756,7	954	0.001,048,2	30.886,890,423,0
905	0.001,105,0	30.083,217,913,0	955	0.001,047,1	30.903,074,280,7
906	0.001,103,8	30.099,833,886,6	956	0.001,046,0	30.919,249,667,5
907	0.001,102,5	30.116,440,692,8	957	0.001,044,9	30.935,416,596,5
908	0.001,101,3	30.133,038,346,6	958	0.001,043,8	30.951,575,081,1
909	0.001,100,1	30.149,626,863,4	959	0.001,042,8	30.967,725,134,4
910	0.001,098,9	30.166,206,258,0	960	0.001,041,6	30.983,866,769,7
911	0.001,097,7	30.182,776,545,6	961	0.001,040,6	31.000,000,000,0
912	0.001,096,5	30.199,337,741,1	962	0.001,039,5	31.016,124,838,5
913	0.001,095,3	30.215,889,859,5	963	0.001,038,4	31.032,241,298,4
914	0.001,094,1	30.232,432,915,7	964	0.001,037,3	31.048,349,392,5
915	0.001,092,9	30.248,966,924,5	965	0.001,036,3	31.064,449,134,0
916	0.001,091,7	30.265,491,900,8	966	0.001,035,2	31.080,540,535,8
917	0.001,090,5	30.282,007,859,5	967	0.001,034,1	31.096,623,610,9
918	0.001,089,3	30.298,514,815,1	968	0.001,033,1	31.112,698,372,2
919	0.001,088,1	30.315,012,782,4	969	0.001,032,0	31.128,764,832,5
920	0.001,087,0	30.331,501,776,2	970	0.001,030,9	31.144,823,004,8
921	0.001,085,8	30.347,981,811,0	971	0.001,029,9	31.160,872,901,8
922	0.001,084,6	30.364,452,901,4	972	0.001,028,8	31.176,914,536,2
923	0.001,083,4	30.380,915,061,9	973	0.001,027,7	31.192,947,921,0
924	0.091,082,3	30.397,368,307,1	974	0.001,026,7	31.208,973,068,7
925	0.001,081,0	30.413,812,651,5	975	0.001,025,6	31.224,989,992,0
926	0.001,079,9	30.430,248,109,4	976	0.001,024,6	31.240,998,703,6
927	0.001,078,7	30.446,674,695,3	977	0.001,023,5	31.256,999,216,2
928	0.001,077,6	30.463,092,423,5	978	0.001,022,5	31.272,991,542,2
929	0.001,076,4	30.479,501,308,3	979	0.001,021,5	31.288,975,694,3
930	0.001,075,3	30.495,901,364,0	980	0.001,020,4	31.304,951,685,0
931	0.001,074,1	30.512,292,604,8	981	0.001,019,4	31.320,919,526,7
932	0.001,073,0	30.528,675,044,9	982	0.001,018,3	31.336,879,232,0
933	0.001,071,8	30.545,048,698,6	983	0.001,017,3	31.352,830,813,2
934	0.001,070,7	30.561,413,579,9	984	0.001,016,3	31.368,774,282,7
935	0.001,069,5	30.577,769,702,8	985	0.001,015,2	31.384,709,653,0
936	0.001,068,4	30.594,117,081,6	986	0.001,014,2	31.400,636,936,2
937	0.001,067,2	30.610,455,730,0	987	0.001,013,2	31.416,556,144,8
938	0.001,066,1	30.626,785,662,2	988	0.001,012,1	31.432,467,291,0
939	0.001,065,0	30.643,106,892,1	989	0.001,011,1	31.448,370,387,0
940	0.001,063,8	30.659,419,433,5	990	0.001,010,1	31.464,265,445,1
941	0.001,062,7	30.675,723,300,4	991	0.001,009,1	31.480,152,477,4
942	0.001,061,6	30.692,018,506,4	992	0.001,008,1	31.496,031,496,0
943	0.001,060,4	30.708,305,065,6	993	0.001,007,0	31.511,902,513,2
944	0.001,059,3	30.724,582,991,5	994	0.001,006,0	31.527,765,540,9
945	0.001,058,2	30.740,852,297,9	995	0.001,005,0	31.543,620,591,2
946	0.001,057,1	30.757,112,998,5	996	0.001,004,0	31.559,467,676,1
947	0.001,056,0	30.773,365,106,9	997	0.001,003,0	31.575,306,807,7
948	0.001,054,9	30.789,608,636,7	998	0.001,002,0	31.591,137,997,9
949	0.001,053,7	30.805,843,601,5	999	0.001,001,0	31.606,961,258,6
950	0.001,052,6	30.822,070,014,8	1000	0.001,	31.622,776,601,7

Dr. HUTTON's *Account of the foregoing Table of the Reciprocals and the Square-Roots of the Natural Numbers* 1, 2, 3, 4, 5, 6, 7, *&c*, *to* 1,000, *given at the end of the Fourth Volume of his Collection of Mathematical Problems and Tracts, intitled* Miſcellanea Mathematica, *publiſhed in Four little Volumes, Duodecimo, in the Year* 1775.

OF the preceeding Table, the uſe is evidently to ſhorten arithmetical calculations, and will appear eminently great to thoſe mathematicians and others who are frequently concerned in ſuch kinds of computations. The ſtructure of the table is evident; the firſt column contains the natural ſeries of numbers from 1 to 1,000, the 2d the reciprocals, and the 3d the ſquare-roots of the ſame numbers, very accurately calculated and printed. Theſe reciprocals and roots are the reſults preſerved of many years occaſional and accidental calculations in various ſubjects: in frequently making ſuch computations, I found that I had often to make diviſions by, and to extract the roots of, the ſame numbers; and as it ſeemed probable that this might be the caſe with me for many years longer, I formed the reſolution of preſerving all ſuch roots and reciprocals as I ſhould occaſionally produce in my calculations, that I might have them always ready on any future occaſion; which I did, by entering them always in a little book, ruled for the purpoſe, till I have at laſt collected to the number of 1,000, as above; and I now publiſh them here in this cheap and eaſy manner, that they may be of like uſe to other perſons as to myſelf. In the numerical calculations of ſuch kinds of problems as have appeared in this Miſcellany and the Diary, the uſe of this table will be found to be very great, becauſe of the frequent diviſions and extractions of roots which are to be made: and the manner and caſes of applying theſe numbers are

are generally evident; only it may be remarked, that the column of reciprocals (which are no other than the decimal values of the quotients refulting from the divifion of unity, or 1, by each of the feveral numbers, from 1 to 1,000), is not only ufeful in fhewing by infpection the quotient when the dividend is unity, but is alfo applied with much advantage in turning many divifions into multiplications, which are much eafier performed than the equivalent divifions. For, if we multiply any propofed dividend by the reciprocal of the divifor (as found in the table,) the product will be the quotient fought; which is the cafe mentioned in p. 54 of my Menfuration, where this table of reciprocals was promifed to be inferted at the end of that Work; but it was then fuppreffed, as the book had been unavoidably extended to fo great a fize, and becaufe it could properly enough be omitted, as being no part of the fubject of the book. This table of reciprocals may alfo be applied to good purpofe in fumming the terms of many converging feriefes, as in the 2d folution of Queft. 106 of this Mifcellany, in which a few of the firft terms are to be found by divifion, and then fummed; for the quotients of fuch divifions are here fhewn by infpection.

The reciprocals are carried on to 7, and the roots to 10 places of decimals, each being put down to the neareft figure in the laft place, that is, when the next figure beyond the laft put down in the table came out a 5, or more, the laft figure was increafed by 1, otherwife not; excepting in the repetends which occurred among the reciprocals, where the real laft figure is always put down. The reciprocals which in the table confift of lefs than feven figures, are thofe which terminate and are complete within that number; fuch as .5 the reciprocal of 2, .25 the reciprocal of 4, &c.

F I N I S